

Minimizing Expected Termination Time in One-Counter Markov Decision Processes

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Abstract. We consider the problem of computing the value and an optimal strategy for minimizing the expected termination time in one-counter Markov decision processes. Since the value may be irrational and an optimal strategy may be rather complicated, we concentrate on the problems of approximating the value up to a given error $\varepsilon > 0$ and computing a finite representation of an ε -optimal strategy. We show that these problems are solvable in exponential time for a given configuration, and we also show that they are computationally hard in the sense that a polynomial-time approximation algorithm cannot exist unless $P=NP$.

1 Introduction

In recent years, a lot of research work has been devoted to the study of stochastic extensions of various automata-theoretic models such as pushdown automata, Petri nets, lossy channel systems, and many others. In this paper we study the class of *one-counter Markov decision processes (OC-MDPs)*, which are infinite-state MDPs [21, 15] generated by finite-state automata operating over a single unbounded counter. Intuitively, an OC-MDP is specified by a finite directed graph \mathcal{A} where the nodes are control states and the edges correspond to transitions between control states. Each control state is either stochastic or non-deterministic, which means that the next edge is chosen either randomly (according to a fixed probability distribution over the outgoing edges) or by a controller. Further, each edge either increments, decrements, or leaves unchanged the current counter value. A *configuration* $q(i)$ of an OC-MDP \mathcal{A} is given by the current control state q and the current counter value i (for technical convenience, we also allow negative counter values, although we are only interested in runs where the counter stays non-negative). The outgoing transitions of $q(i)$ are determined by the edges of \mathcal{A} in the natural way.

Previous works on OC-MDPs [5, 3, 4] considered mainly the objective of *maximizing/minimizing termination probability*. We say that a run initiated in a configuration $q(i)$ *terminates* if it visits a configuration with zero counter. The goal of the controller is to play so that the probability of all terminating runs is maximized (or minimized).

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In this paper, we study a related objective of *minimizing the expected termination time*. Formally, we define a random variable T over the runs of \mathcal{A} such that $T(\omega)$ is equal either to ∞ (if the run ω is non-terminating) or to the number of transitions need to reach a configuration with zero counter (if ω is terminating). The goal of the controller is to minimize the expectation $\mathbb{E}(T)$. The *value* of $q(i)$ is the infimum of $\mathbb{E}(T)$ over all strategies. It is easy to see that the controller has a memoryless deterministic strategy which is optimal (i.e., achieves the value) in every configuration. However, since OC-MDPs have infinitely many configurations, this does not imply that an optimal strategy is finitely representable and computable. Further, the value itself can be irrational. Therefore, we concentrate on the problem of *approximating* the value of a given configuration up to a given (absolute or relative) error $\varepsilon > 0$, and computing a strategy which is ε -optimal (in both absolute and relative sense). Our main results can be summarized as follows:

- ***The value and optimal strategy can be effectively approximated up to a given relative/absolute error in exponential time.*** More precisely, we show that given a OC-MDP \mathcal{A} , a configuration $q(i)$ of \mathcal{A} where $i \geq 0$, and $\varepsilon > 0$, the value of $q(i)$ up to the (relative or absolute) error ε is computable in time exponential in the encoding size of \mathcal{A} , i , and ε , where all numerical constants are represented as fractions of binary numbers. Further, there is a history-dependent deterministic strategy σ computable in exponential time such that the absolute/relative difference between the value of $q(i)$ and the outcome of σ in $q(i)$ is bounded by ε .
- ***The value is not approximable in polynomial time unless $P=NP$.*** This hardness result holds even if we restrict ourselves to configurations with counter value equal to 1 and to OC-MDPs where every outgoing edge of a stochastic control state has probability $1/2$. The result is valid for absolute as well as relative approximation.

Let us sketch the basic ideas behind these results. The upper bounds are obtained in two steps. In the first step (Section 3.1), we analyze the special case when the underlying graph of \mathcal{A} is strongly connected. We show that minimizing the expected termination time is closely related to minimizing the expected increase of the counter per transition, at least for large counter values. We start by computing the minimal expected increase of the counter per transition (denoted by \bar{x}) achievable by the controller, and the associated strategy σ . This is done by standard linear programming techniques developed for optimizing the long-run average reward in finite-state MDPs (see, e.g., [21]) applied to the underlying finite graph of \mathcal{A} . Note that σ depends only on the current control state and ignores the current counter value (we say that σ is *counterless*). Further, the encoding size of \bar{x} is *polynomial* in $\|\mathcal{A}\|$. Then, we distinguish two cases.

Case (A), $\bar{x} \geq 0$. Then the counter does not have a tendency to decrease *regardless* of the controller's strategy, and the expected termination time value is infinite in all configurations $q(i)$ such that $i \geq |Q|$, where Q is the set of control states of \mathcal{A} (see Proposition 5. A). For the finitely many remaining configurations, we can compute the value and optimal strategy precisely by standard methods for finite-state MDPs.

Case (B), $\bar{x} < 0$. Then, one intuitively expects that applying the strategy σ in an initial configuration $q(i)$ yields the expected termination time about $i/|\bar{x}|$. Actually, this is *almost* correct; we show (Proposition 5. B.2) that this expectation is bounded by $(i + U)/|\bar{x}|$, where $U \geq 0$ is a constant depending only on \mathcal{A} whose size is at most

exponential in $\|\mathcal{A}\|$. Further, we show that an *arbitrary* strategy π applied to $q(i)$ yields the expected termination time *at least* $(i - V)/\|\bar{x}\|$, where $V \geq 0$ is a constant depending only on \mathcal{A} whose size is at most exponential in $\|\mathcal{A}\|$ (Proposition 5. B.1). In particular, this applies to the *optimal* strategy π^* for minimizing the expected termination time. Hence, π^* can be more efficient than σ , but the difference between their outcomes is bounded by a constant which depends only on \mathcal{A} and is at most exponential in $\|\mathcal{A}\|$. We proceed by computing a sufficiently large k so that the probability of increasing the counter to $i + k$ by a run initiated in $q(i)$ is inevitably (i.e., under any optimal strategy) so small that the controller can safely switch to the strategy σ when the counter reaches the value $i + k$. Then, we construct a *finite-state* MDP \mathcal{M} and a reward function f over its transitions such that

- the states are all configurations $p(j)$ where $0 \leq j \leq i + k$;
- all states with counter values less than $i + k$ “inherit” their transitions from \mathcal{A} ; configurations of the form $p(i + k)$ have only self-loops;
- the self-loops on configurations where the counter equals 0 or $i + k$ have zero reward, transitions leading to configurations where the counter equals $i + k$ have reward $(i + k + U)/\|\bar{x}\|$, and the other transitions have reward 1.

In this finite-state MDP \mathcal{M} , we compute an optimal memoryless deterministic strategy ϱ for the total accumulated reward objective specified by f . Then, we consider another strategy $\hat{\sigma}$ for $q(i)$ which behaves like ϱ until the point when the counter reaches $i + k$, and from that point on it behaves like σ . It turns out that the absolute as well as relative difference between the outcome of $\hat{\sigma}$ in $q(i)$ and the value of $q(i)$ is bounded by ε , and hence $\hat{\sigma}$ is the desired ε -optimal strategy.

In the general case when \mathcal{A} is not necessarily strongly connected (see Section 3.2), we have to solve additional difficulties. Intuitively, we split the graph of \mathcal{A} into maximal end components (MECs), where each MEC can be seen as a strongly connected OC-MDP and analyzed by the techniques discussed above. In particular, for every MEC C we compute the associated \bar{x}_C (see above). Then, we consider a strategy which tries to reach a MEC as quickly as possible so that the expected value of the fraction $1/\|\bar{x}_C\|$ is *minimal*. After reaching a target MEC, the strategy starts to behave as the strategy σ discussed above. It turns out that this particular strategy cannot be much worse than the optimal strategy (a proof of this claim requires new observations), and the rest of the argument is similar as in the strongly connected case.

The lower bound, i.e., the result saying that the value cannot be efficiently approximated unless $P=NP$ (see Section 4), seems to be the first result of this kind for OC-MDPs. Here we combine the technique of encoding propositional assignments presented in [19] (see also [17]) with some new gadgets constructed specifically for this proof (let us note that we did not manage to improve the presented lower bound to PSPACE by adapting other known techniques [16, 22, 18]). As a byproduct, our proof also reveals that the optimal strategy for minimizing the expected termination time *cannot* ignore the precise counter value, even if the counter becomes very large. In our example, the (only) optimal strategy is *eventually periodic* in the sense that for a sufficiently large counter value i , it is only “ i modulo c ” which matters, where c is a fixed (exponentially large) constant. The question whether there *always* exists an optimal

eventually periodic strategy is left open. Another open question is whether our results can be extended to stochastic games over one-counter automata.

Related work: One-counter automata can also be seen as pushdown automata with one letter stack alphabet. Stochastic games and MPDs generated by pushdown automata and stateless pushdown automata (also known as BPA) with termination and reachability objectives have been studied in [13, 14, 6, 7]. To the best of our knowledge, the only prior work on the expected termination time (or, more generally, total accumulated reward) objective for a class of infinite-state MDPs or stochastic games is [11], where this problem is studied for stochastic BPA games. The termination objective for one-counter MDPs and games has been examined in [5, 3, 4], where it was shown (among other things) that the equilibrium termination probability (i.e., the termination value) can be approximated up to a given precision in exponential time, but no lower bound was provided. The games over one-counter automata are also known as “energy games” [9, 10]. Intuitively, the counter is used to model the amount of currently available energy, and the aim of the controller is to optimize the energy consumptions. Finally, let us note that OC-MDPs can be seen as discrete-time Quasi-Birth-Death Processes (QBDs, see, e.g., [20, 12]) extended with a control. Hence, the theory of one-counter MDPs and games is closely related to queuing theory, where QBDs are considered as a fundamental model.

2 Preliminaries

Given a set A , we use $|A|$ to denote the cardinality of A . We also write $|x|$ to denote the absolute value of a given $x \in \mathbb{R}$, but this should not cause any confusions. The encoding size of a given object B is denoted by $\|B\|$. The set of integers is denoted by \mathbb{Z} , and the set of positive integers by \mathbb{N} .

We assume familiarity with basic notions of probability theory. In particular, we call a probability distribution f over a discrete set A *positive* if $f(a) > 0$ for all $a \in A$, and *Dirac* if $f(a) = 1$ for some $a \in A$.

Definition 1 (MDP). A Markov decision process (MDP) is a tuple $\mathcal{M} = (S, (S_0, S_1), \rightsquigarrow, \text{Prob})$, consisting of a countable set of states S partitioned into the sets S_0 and S_1 of stochastic and non-deterministic states, respectively. The edge relation $\rightsquigarrow \subseteq S \times S$ is total, i.e., for every $r \in S$ there is $s \in S$ such that $r \rightsquigarrow s$. Finally, Prob assigns to every $s \in S_0$ a positive probability distribution over its outgoing edges.

A *finite path* is a sequence $w = s_0 s_1 \cdots s_n$ of states such that $s_i \rightsquigarrow s_{i+1}$ for all $0 \leq i < n$. We write $\text{len}(w) = n$ for the length of the path. A *run* is an infinite sequence ω of states such that every finite prefix of ω is a path. For a finite path, w , we denote by $\text{Run}(w)$ the set of runs having w as a prefix. These generate the standard σ -algebra on the set of runs.

Definition 2 (OC-MDP). A one-counter MDP (OC-MDP) is a tuple $\mathcal{A} = (Q, (Q_0, Q_1), \delta, P)$, where Q is a finite non-empty set of control states partitioned into stochastic and non-deterministic states (as in the case of MDPs), $\delta \subseteq Q \times \{+1, 0, -1\} \times Q$ is a set of transition rules such that $\delta(q) := \{(q, i, r) \in \delta\} \neq \emptyset$ for all $q \in Q$, and

$P = \{P_q\}_{q \in Q_0}$ where P_q is a positive rational probability distribution over $\delta(q)$ for all $q \in Q_0$.

In the rest of this paper we often write $q \xrightarrow{i} r$ to indicate that $(q, i, r) \in \delta$, and $q \xrightarrow{i, x} r$ to indicate that $(q, i, r) \in \delta$, q is stochastic, and $P_q(q, i, r) = x$. Without restrictions, we assume that for each pair $q, r \in Q$ there is at most one i such that $(q, i, r) \in \delta$. The encoding size of \mathcal{A} is denoted by $\|\mathcal{A}\|$, where all numerical constants are encoded as fractions of binary numbers. The set of all configurations is $C := \{q(i) \mid q \in Q, i \in \mathbb{Z}\}$.

To \mathcal{A} we associate an infinite-state MDP $\mathcal{M}_{\mathcal{A}}^{\infty} = (C, (C_0, C_1), \sim, Prob)$, where the partition of C is defined by $q(i) \in C_0$ iff $q \in Q_0$, and similarly for C_1 . The edges are defined by $q(i) \sim r(j)$ iff $(q, j - i, r) \in \delta$. The probability assignment $Prob$ is derived naturally from P .

By forgetting the counter values, the OC-MDP \mathcal{A} also defines a finite-state MDP $\mathcal{M}_{\mathcal{A}} = (Q, (Q_0, Q_1), \sim, Prob')$. Here $q \sim r$ iff $(q, i, r) \in \delta$ for some i , and $Prob'$ is derived in the obvious way from P by forgetting the counter changes.

Strategies and Probability. Let \mathcal{M} be an MDP. A *history* is a finite path in \mathcal{M} , and a *strategy* (or *policy*) is a function assigning to each history ending in a state from S_1 a distribution on edges leaving the last state of the history. A strategy σ is *pure* (or *deterministic*) if it always assigns 1 to one edge and 0 to the others, and *memoryless* if $\sigma(w) = \sigma(s)$ where s is the last state of a history w .

Now consider some OC-MDP \mathcal{A} . A strategy σ over the histories in $\mathcal{M}_{\mathcal{A}}^{\infty}$ is *counterless* if it is memoryless and $\sigma(q(i)) = \sigma(q(j))$ for all i, j . Observe that every strategy σ for $\mathcal{M}_{\mathcal{A}}^{\infty}$ gives a unique strategy σ' for $\mathcal{M}_{\mathcal{A}}$ which just forgets the counter values in the history and plays as σ . This correspondence is bijective when restricted to memoryless strategies in $\mathcal{M}_{\mathcal{A}}$ and counterless strategies in $\mathcal{M}_{\mathcal{A}}^{\infty}$, and it is used implicitly throughout the paper.

Fixing a strategy σ and an initial state s , we obtain in a standard way a probability measure $\mathbb{P}_s^{\sigma}(\cdot)$ on the subspace of runs starting in s . For MDPs of the form $\mathcal{M}_{\mathcal{A}}^{\infty}$ for some OC-MDP \mathcal{A} , we consider two sequences of random variables, $\{C^{(i)}\}_{i \geq 0}$ and $\{S^{(i)}\}_{i \geq 0}$, returning the current counter value and the current control state after completing i transitions.

Termination Time in OC-MDPs. Let \mathcal{A} be a OC-MDP. A run ω in $\mathcal{M}_{\mathcal{A}}^{\infty}$ *terminates* if $\omega(j) = q(0)$ for some $j \geq 0$ and $q \in Q$. The associated *termination time*, denoted by $T(\omega)$, is the least j such that $\omega(j) = q(0)$ for some $q \in Q$. If there is no such j , we put $T(\omega) = \infty$, where the symbol ∞ denotes the “infinite amount” with the standard conventions, i.e., $c < \infty$ and $\infty + c = \infty + \infty = \infty \cdot d = \infty$ for arbitrary real numbers c, d where $d > 0$.

For every strategy σ and a configuration $q(i)$, we use $\mathbb{E}^{\sigma} q(i)$ to denote the expected value of T in the probability space of all runs initiated in $q(i)$ where $\mathbb{P}_{q(i)}^{\sigma}(\cdot)$ is the underlying probability measure. The *value* of a given configuration $q(i)$ is defined by $\text{Val}(q(i)) := \inf_{\sigma} \mathbb{E}^{\sigma} q(i)$. Let $\varepsilon \geq 0$ and $i \geq 1$. We say that a constant v approximates $\text{Val}(q(i))$ up to the absolute or relative error ε if $|\text{Val}(q(i)) - v| \leq \varepsilon$ or $|\text{Val}(q(i)) - v| / \text{Val}(q(i)) \leq \varepsilon$, respectively. Note that if v approximates $\text{Val}(q(i))$ up to the absolute error ε , then it also approximates $\text{Val}(q(i))$ up to the relative error ε because $\text{Val}(q(i)) \geq 1$. A strategy σ is (absolutely or relatively) ε -*optimal* if $\mathbb{E}^{\sigma} q(i)$ approximates $\text{Val}(q(i))$ up to the (absolute or relative) error ε . A 0-optimal strategy is called *optimal*.

maximize x , subject to

$$\begin{aligned} z_q &\leq -x + k + z_r && \text{for all } q \in Q_1 \text{ and } (q, k, r) \in \delta, \\ z_q &\leq -x + \sum_{(q,k,r) \in \delta} P_q((q, k, r)) \cdot (k + z_r) && \text{for all } q \in Q_0, \end{aligned}$$

Fig. 1. The linear program \mathcal{L} over x and $z_q, q \in Q$.

It is easy to see that there is a memoryless deterministic strategy σ in $\mathcal{M}_{\mathcal{A}}^\infty$ which is optimal in every configuration of $\mathcal{M}_{\mathcal{A}}^\infty$. First, observe that for all $q \in Q_0, q' \in Q_1$, and $i \neq 0$ we have that

$$\begin{aligned} \text{Val}(q(i)) &= 1 + \sum_{q(i) \leadsto r(j)} x \cdot \text{Val}(r(j)) \\ \text{Val}(q'(i)) &= 1 + \min\{\text{Val}(r(j)) \mid q'(i) \leadsto r(j)\}. \end{aligned}$$

We put $\sigma(q(i)) = r(j)$ where $q(i) \leadsto r(j)$ and $\text{Val}(q(i)) = 1 + r(j)$ (if there are several candidates for $r(j)$, any of them can be chosen). Now we can easily verify that σ is indeed optimal in every configuration.

3 Upper Bounds

The goal of this section is to prove the following:

Theorem 3. *Let \mathcal{A} be a OC-MDP, $q(i)$ a configuration of \mathcal{A} where $i \geq 0$, and $\varepsilon > 0$.*

1. *The problem whether $\text{Val}(q(i)) = \infty$ is decidable in polynomial time.*
2. *There is an algorithm that computes a rational number v such that $|\text{Val}(q(i)) - v| \leq \varepsilon$, and a strategy σ that is absolutely ε -optimal starting in $q(i)$. The algorithm runs in time exponential in $\|\mathcal{A}\|$ and polynomial in i and $1/\varepsilon$. (Note that v then approximates $\text{Val}(q(i))$ also up to the relative error ε , and σ is also relatively ε -optimal in $q(i)$).*

For the rest of this section, we fix an OC-MDP $\mathcal{A} = (Q, (Q_0, Q_1), \delta, P)$. First, we prove Theorem 3 under the assumption that $\mathcal{M}_{\mathcal{A}}$ is *strongly connected* (Section 3.1). A generalization to arbitrary OC-MDP is then given in Section 3.2.

3.1 Strongly connected OC-MDP

Let us assume that $\mathcal{M}_{\mathcal{A}}$ is strongly connected, i.e., for all $p, q \in Q$ there is a finite path from p to q in $\mathcal{M}_{\mathcal{A}}$. Consider the linear program of Figure 1. Intuitively, the variable x encodes a lower bound on the long-run trend of the counter value. More precisely, the maximal value of x corresponds to the *minimal* long-run average change in the counter value achievable by some strategy. The program corresponds to the one used for optimizing the long-run average reward in Sections 8.8 and 9.5 of [21], and hence we know it has a solution.

Lemma 4 ([21]). *There is a rational solution $(\bar{x}, (\bar{z}_q)_{q \in Q}) \in \mathbb{Q}^{|Q|+1}$ to \mathcal{L} , and the encoding size³ of the solution is polynomial in $\|\mathcal{A}\|$.*

³ Recall that rational numbers are represented as fractions of binary numbers.

Note that $\bar{x} \geq -1$, because for any fixed $x \leq -1$ the program \mathcal{L} trivially has a feasible solution. Further, we put $V := \max_{q \in Q} \bar{z}_q - \min_{q \in Q} \bar{z}_q$. Observe that $V \in \exp(\|\mathcal{A}\|^{O(1)})$ and V is computable in time polynomial in $\|\mathcal{A}\|$.

Proposition 5. *Let $(\bar{x}, (\bar{z}_q)_{q \in Q})$ be a solution of \mathcal{L} .*

- (A) *If $\bar{x} \geq 0$, then $\text{Val}(q(i)) = \infty$ for all $q \in Q$ and $i \geq |Q|$.*
- (B) *If $\bar{x} < 0$, then the following holds:*
 - (B.1) *For every strategy π and all $q \in Q$, $i \geq 0$ we have that $\mathbb{E}^\pi q(i) \geq (i - V)/|\bar{x}|$.*
 - (B.2) *There is a counterless strategy σ and a number $U \in \exp(\|\mathcal{A}\|^{O(1)})$ such that for all $q \in Q$, $i \geq 0$ we have that $\mathbb{E}^\sigma q(i) \leq (i + U)/|\bar{x}|$. Moreover, σ and U are computable in time polynomial in $\|\mathcal{A}\|$.*

First, let us realize that Proposition 5 implies Theorem 3. To see this, we consider the cases $\bar{x} \geq 0$ and $\bar{x} < 0$ separately. In both cases, we resort to analyzing a finite-state MDP \mathcal{G}_K , where K is a suitable natural number, obtained by restricting $\mathcal{M}_{\mathcal{A}}^\infty$ to configurations with counter value at most K , and by substituting all transitions leaving each $p(K)$ with a self-loop of the form $p(K) \leadsto p(K)$.

First, let us assume that $\bar{x} \geq 0$. By Proposition 5 (A), we have that $\text{Val}(q(i)) = \infty$ for all $q \in Q$ and $i \geq |Q|$. Hence, it remains to approximate the value and compute ε -optimal strategy for all configurations $q(i)$ where $i \leq |Q|$. Actually, we can even compute these values precisely and construct a strategy $\hat{\sigma}$ which is optimal in each such $q(i)$. This is achieved simply by considering the finite-state MDP $\mathcal{G}_{|Q|}$ and solving the objective of minimizing the expected number of transitions needed to reach a state of the form $p(0)$, which can be done by standard methods in time polynomial in $\|\mathcal{A}\|$.

If $\bar{x} < 0$, we argue as follows. The strategy σ of Proposition 5 (B.2) is not necessarily ε -optimal in $q(i)$, so we cannot use it directly. To overcome this problem, consider an *optimal* strategy π^* in $q(i)$, and let x_ℓ be the probability that a run initiated in $q(i)$ (under the strategy π^*) visits a configuration of the form $r(i + \ell)$. Obviously, $x_\ell \cdot \min_{r \in Q} \{\mathbb{E}^{\pi^*} r(i + \ell)\} \leq \mathbb{E}^{\pi^*} q(i)$, because otherwise π^* would not be optimal in $q(i)$. Using the lower/upper bounds for $\mathbb{E}^{\pi^*} r(i + \ell)$ and $\mathbb{E}^\sigma q(i)$ given in Proposition 5 (B), we obtain $x_\ell \leq (i + U)/(i + \ell - V)$. Then, we compute $k \in \mathbb{N}$ such that

$$x_k \cdot \left(\max_{r \in Q} \left\{ (i + k + U)/|\bar{x}| - \mathbb{E}^{\pi^*} r(i + k) \right\} \right) \leq \varepsilon$$

A simple computation reveals that it suffices to put

$$k \geq \frac{(i + U)(U + V)}{\varepsilon|\bar{x}|} + V - i$$

Now, consider \mathcal{G}_{i+k} , and let f be a reward function over the transitions of \mathcal{G}_{i+k} such that the loops on configurations where the counter equals 0 or $i + k$ have zero reward, a transition leading to a state $r(i + k)$ has reward $(i + k + U)/|\bar{x}|$, and all of the remaining transitions have reward 1. Now we solve the finite-state MDP \mathcal{G}_{i+k} with the objective of minimizing the total accumulated reward. Note that an optimal strategy ϱ in \mathcal{G}_{i+k} is computable in time polynomial in the size of \mathcal{G}_{i+k} [21]. Then, we define the corresponding strategy $\hat{\sigma}$ in $\mathcal{M}_{\mathcal{A}}^\infty$, which behaves like ϱ until the counter reaches $i + k$, and from

that point on it behaves like the counterless strategy σ . It is easy to see that $\hat{\sigma}$ is indeed ε -optimal in $q(i)$.

Proof of Proposition 5. Similarly as in [4], we use the solution $(\bar{x}, (\bar{z}_q)_{q \in Q}) \in \mathbb{Q}^{|Q|+1}$ of \mathcal{L} to define a suitable submartingale, which is then used to derive the required bounds. In [4], Azuma's inequality was applied to the submartingale to prove exponential tail bounds for termination probability. In this paper, we need to use the optional stopping theorem rather than Azuma's inequality, and therefore we need to define the submartingale relative to a suitable filtration so that we can introduce an appropriate stopping time (without the filtration, the stopping time would have to depend just on numerical values returned by the martingale, which does not suit our purposes).

Recall the random variables $\{C^{(i)}\}_{i \geq 0}$ and $\{S^{(i)}\}_{i \geq 0}$ returning the height of the counter, and the control state after completing i transitions, respectively. Given the solution $(\bar{x}, (\bar{z}_q)_{q \in Q}) \in \mathbb{Q}^{|Q|+1}$ from Lemma 4, we define a sequence of random variables $\{m^{(i)}\}_{i \geq 0}$ by setting

$$m^{(i)} := \begin{cases} C^{(i)} + \bar{z}_{S^{(i)}} - i \cdot \bar{x} & \text{if } C^{(j)} > 0 \text{ for all } j, 0 \leq j < i, \\ m^{(i-1)} & \text{otherwise.} \end{cases}$$

Note that for every history u of length i and every $0 \leq j \leq i$, the random variable $m^{(j)}$ returns the same value for every $\omega \in \text{Run}(u)$. The same holds for variables $S^{(j)}$ and $C^{(j)}$. We will denote these common values $m^{(j)}(u)$, $S^{(j)}(u)$ and $C^{(j)}(u)$, respectively. Using the same arguments as in Lemma 3 of [4], one may show that for every history u of length i we have $\mathbb{E}(m^{(i+1)} \mid \text{Run}(u)) \geq m^{(i)}(u)$. This shows that $\{m^{(i)}\}_{i \geq 0}$ is a *submartingale relative to the filtration* $\{\mathcal{F}_i\}_{i \geq 0}$, where for each $i \geq 0$ the σ -algebra \mathcal{F}_i is the σ -algebra generated by all $\text{Run}(u)$ where $\text{len}(u) = i$. Intuitively, this means that value $m^{(i)}(\omega)$ is uniquely determined by prefix of ω of length i and that the process $\{m^{(i)}\}_{i \geq 0}$ has nonnegative average change. For relevant definitions of (sub)martingales see, e.g., [23]. Another important observation is that $|m^{(i+1)} - m^{(i)}| \leq 1 + \bar{z} + V$ for every $i \geq 0$, i.e., the differences of the submartingale are bounded.

Lemma 6. *Under an arbitrary strategy τ and with an arbitrary initial configuration $q(j)$ where $j \geq 0$, the process $\{m^{(i)}\}_{i \geq 0}$ is a submartingale (relative to the filtration $\{\mathcal{F}_i\}_{i \geq 0}$) with bounded differences.*

Part (A) of Proposition 5. This part can be proved by a routine application of the optional stopping theorem to the martingale $\{m^{(i)}\}_{i \geq 0}$. Let $\bar{z}_{\max} := \max_{q \in Q} \bar{z}_q$, and consider a configuration $p(\ell)$ where $\ell + \bar{z}_r > \bar{z}_{\max}$. Let σ be a strategy which is optimal in every configuration. Assume, for the sake of contradiction, that $\text{Val}(p(\ell)) < \infty$.

Let us fix $k \in \mathbb{N}$ such that $\ell + \bar{z}_r < \bar{z}_{\max} + k$ and define a stopping time τ which returns the first point in time in which either $m^{(\tau)} \geq \bar{z}_{\max} + k$, or $m^{(\tau)} \leq \bar{z}_{\max}$. To apply the optional stopping theorem, we need to show that the expectation of τ is finite.

We argue that every configuration $q(i)$ with $i \geq 1$ satisfies the following: under the optimal strategy σ , a configuration with counter height $i - 1$ is reachable from $q(i)$ in at most $|Q|^2$ steps (i.e., with a bounded probability). To see this, realize that for every configuration $r(j)$ there is a successor, say $r'(j')$, such that $\text{Val}(r(j)) > \text{Val}(r'(j'))$. Now consider a run w initiated in $q(i)$ obtained by subsequently choosing successors with smaller and smaller values. Note that whenever $w(j)$ and $w(j')$ with $j < j'$ have the

same control state, the counter height of $w(j')$ must be strictly smaller than the one of $w(j)$ because otherwise the strategy σ could be improved (it suffices to behave in $w(j)$ as in $w(j')$). It follows that there must be $k \leq |Q|^2$ such that the counter height of $w(k)$ is $i - 1$. From this we obtain that the expected value of τ is finite because the probability of terminating from any configuration with bounded counter height is bounded from zero. Now we apply the optional stopping theorem and obtain $\mathbb{P}_{p(\ell)}^\sigma(m^{(\tau)} \geq \bar{z}_{\max} + k) \geq c/(k+d)$ for suitable constants $c, d > 0$. As $m^{(\tau)} \geq \bar{z}_{\max} + k$ implies $C^{(\tau)} \geq k$, we obtain that

$$\mathbb{P}_{p(\ell)}^\sigma(T \geq k) \geq \mathbb{P}_{p(\ell)}^\sigma(C^{(\tau)} \geq k) \geq \mathbb{P}_{p(\ell)}^\sigma(m^{(\tau)} \geq \bar{z}_{\max} + k) \geq \frac{c}{k+d}$$

and thus

$$\mathbb{E}^\sigma p(\ell) = \sum_{k=1}^{\infty} \mathbb{P}_{p(\ell)}^\sigma(T \geq k) \geq \sum_{k=1}^{\infty} \frac{c}{k+d} = \infty$$

which contradicts our assumption that σ is optimal and $\text{Val}(p(\ell)) < \infty$.

It remains to show that $\text{Val}(p(\ell)) = \infty$ even for $\ell = |Q|$. This follows from the following simple observation:

Lemma 7. *For all $q \in Q$ and $i \geq |Q|$ we have that $\text{Val}(q(i)) < \infty$ iff $\text{Val}(q(|Q|)) < \infty$.*

The “only if” direction of Lemma 7 is trivial. For the other direction, let \mathcal{B}_k denote the set of all $p \in Q$ such that $\text{Val}(p(k)) < \infty$. Clearly, $\mathcal{B}_0 = Q$, $\mathcal{B}_k \subseteq \mathcal{B}_{k-1}$, and one can easily verify that $\mathcal{B}_k = \mathcal{B}_{k+1}$ implies $\mathcal{B}_k = \mathcal{B}_{k+\ell}$ for all $\ell \geq 0$. Hence, $\mathcal{B}_{|Q|} = \mathcal{B}_{|Q|+\ell}$ for all ℓ . Note that Lemma 7 holds for general OC-MDPs (i.e., we do not need to assume that $\mathcal{M}_{\mathcal{A}}$ is strongly connected).

Part (B1) of Proposition 5. Let π be a strategy and $q(i)$ a configuration where $i \geq 0$. If $\mathbb{E}^\pi q(i) = \infty$, we are done. Now assume $\mathbb{E}^\pi q(i) < \infty$. Observe that for every $k \geq 0$ and every run ω , the membership of ω into $\{T \leq k\}$ depends only on the finite prefix of ω of length k . This means that T is a stopping time relative to filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Since $\mathbb{E}^\pi q(i) < \infty$ and the submartingale $\{m^{(n)}\}_{n \geq 0}$ has bounded differences, we can apply the optional stopping theorem and obtain $\mathbb{E}^\pi(m^{(0)}) \leq \mathbb{E}^\pi(m^{(T)})$. But $\mathbb{E}^\pi(m^{(0)}) = i + \bar{z}_q$ and $\mathbb{E}^\pi(m^{(T)}) = \mathbb{E}^\pi \bar{z}_{S^{(T)}} + \mathbb{E}^\pi q(i) \cdot |\bar{x}|$. Thus, we get $\mathbb{E}^\pi q(i) \geq (i + \bar{z}_q - \mathbb{E}^\pi \bar{z}_{S^{(T)}})/|\bar{x}| \geq (i - V)/|\bar{x}|$.

Part (B2) of Proposition 5. First we show how to construct the desired strategy σ . Recall again the linear program \mathcal{L} of Figure 1. We have already shown that this program has an optimal solution $(\bar{x}, (\bar{z}_q)_{q \in Q}) \in \mathbb{Q}^{|Q|+1}$, and we assume that $\bar{x} < 0$. By the strong duality theorem, this means that the linear program dual to \mathcal{L} also has a feasible solution $(\bar{y}_q)_{q \in Q_0}, (\bar{y}_{(q,i,q')})_{q \in Q_1, (q,i,q') \in \delta}$. Let

$$D = \{q \in Q_0 \mid \bar{y}_q > 0\} \cup \{q \in Q_1 \mid \bar{y}_{(q,i,q')} > 0 \text{ for some } (q,i,q') \in \delta\}.$$

By Corollary 8.8.8 of [21], the solution $(\bar{y}_q)_{q \in Q_0}, (\bar{y}_{(q,i,q')})_{q \in Q_1, (q,i,q') \in \delta}$ can be chosen so that for every $q \in Q_1$ there is at most one transition (q,i,q') with $\bar{y}_{(q,i,q')} > 0$. Following the construction given in Section 8.8 of [21], we define a counterless deterministic strategy σ such that

- in a state $q \in D \cap Q_1$, the strategy σ selects the transition (q, i, q') with $\bar{y}_{(q,i,q')} > 0$;
- in the states outside D , the strategy σ behaves like an optimal strategy for the objective of reaching the set D .

Clearly, the strategy σ is computable in time polynomial in $\|\mathcal{A}\|$. To show that σ indeed satisfies Part (B.2) of Proposition 5, we need to prove a series of auxiliary inequalities, which can be found in Appendix A.1.

3.2 General OC-MDP

In this section we prove Theorem 3 for general OC-MDPs, i.e., we drop the assumption that $\mathcal{M}_{\mathcal{A}}$ is strongly connected. We say that $C \subseteq Q$ is an *end component* of \mathcal{A} if C is strongly connected and for every $p \in C \cap Q_0$ we have that $\{q \in Q \mid p \rightsquigarrow q\} \subseteq C$. A *maximal end component (MEC)* of \mathcal{A} is an end component of \mathcal{A} which is maximal w.r.t. set inclusion. The set of all MECs of \mathcal{A} is denoted by $\text{MEC}(\mathcal{A})$. Every $C \in \text{MEC}(\mathcal{A})$ determines a strongly connected OC-MDP $\mathcal{A}_C = (C, (C \cap Q_0, C \cap Q_1), \delta \cap (C \times \{+1, 0, -1\} \times C), \{P_q\}_{q \in C \cap Q_0})$. Hence, we may apply Proposition 5 to \mathcal{A}_C , and we use \bar{x}_C and V_C to denote the constants of Proposition 5 computed for \mathcal{A}_C .

Part 1. of Theorem 3. We show how to compute, in time polynomial in $\|\mathcal{A}\|$, the set $Q_{\text{fin}} = \{p \in Q \mid \text{Val}(p(k)) < \infty \text{ for all } k \geq 0\}$. From this we easily obtain Part 1. of Theorem 3, because for every configuration $q(i)$ where $i \geq 0$ we have the following:

- if $i \geq |Q|$, then $\text{Val}(q(i)) < \infty$ iff $q \in Q_{\text{fin}}$ (see Lemma 7);
- if $i < |Q|$, then $\text{Val}(q(i)) < \infty$ iff the set $\{p(0) \mid p \in Q\} \cup \{p(|Q|) \mid p \in Q_{\text{fin}}\}$ can be reached from $q(i)$ with probability 1 in the finite-state MDP $\mathcal{G}_{|Q|}$ defined in Section 3.1 (here we again use Lemma 7).

So, it suffices to show how to compute the set Q_{fin} in polynomial time.

Proposition 8. *Let $Q_{<0}$ be the set of all states from which the set $H = \{q \in Q \mid q \text{ belongs to a MEC } C \text{ satisfying } \bar{x}_C < 0\}$ is reachable with probability 1. Then $Q_{\text{fin}} = Q_{<0}$. Moreover, the membership to $Q_{<0}$ is decidable in time polynomial in $\|\mathcal{A}\|$.*

Part 2. of Theorem 3. First, we generalize Part (B) of Proposition 5 into the following:

Proposition 9. *For every $q \in Q_{\text{fin}}$ there is a number t_q computable in time polynomial in $\|\mathcal{A}\|$ such that $-1 \leq t_q < 0$, $1/|t_q| \in \exp(\|\mathcal{A}\|^{O(1)})$, and the following holds:*

- There is a counterless strategy σ and a number $U \in \exp(\|\mathcal{A}\|^{O(1)})$ such that for every configuration $q(i)$ where $q \in Q_{\text{fin}}$ and $i \geq 0$ we have that $\mathbb{E}^\sigma q(i) \leq i/|t_q| + U$. Moreover, both σ and U are computable in time polynomial in $\|\mathcal{A}\|$.*
- There is a number $L \in \exp(\|\mathcal{A}\|^{O(1)})$ such that for every strategy π and every configuration $q(i)$ where $i \geq |Q|$ we have that $\mathbb{E}^\pi \geq i/|t_q| - L$. Moreover, L is computable in time polynomial in $\|\mathcal{A}\|$.*

Once the Proposition 9 is proved, we can compute an ε -optimal strategy for an arbitrary configuration $q(i)$ where $q \in Q_{fin}$ and $i \geq |Q|$ in exactly the same way (and with the same complexity) as in the strongly connected case. Actually, it can also be used to compute the approximate values and ε -optimal strategies for configurations $q(j)$ such that $q \notin Q_{fin}$ or $1 \leq j < |Q|$. Observe that

- if $q \notin Q_{fin}$ and $j \geq |Q|$, the value is infinite by Part 1;
- otherwise, we construct the finite-state MDP $\mathcal{G}_{|Q|}$ (see Section 3.1) where the loops on configurations with counter value 0 have reward 0, the loops on configurations of the form $r(|Q|)$ have reward 0 or 1, depending on whether $r \in Q_{fin}$ or not, transitions leading to $r(|Q|)$ where $r \in Q_{fin}$ are rewarded with some ε -approximation of $\text{Val}(r(|Q|))$, and all other transitions have reward 1. The reward function can be computed in time exponential in $\|\mathcal{A}\|$ by Proposition 9, and the minimal total accumulated reward from $q(j)$ in $\mathcal{G}_{|Q|}$, which can be computed by standard algorithms, is an ε -approximation of $\text{Val}(q(j))$. The corresponding ε -optimal strategy can be computed in the obvious way.

The proof of Propositions 8 and 9 can be found in Appendices A.2 and A.3, respectively.

4 Lower Bounds

In this section, we show that approximating $\text{Val}(q(i))$ is computationally hard, even if $i = 1$ and the edge probabilities in the underlying OC-MDP are all equal to $1/2$. More precisely, we prove the following:

Theorem 10. *The value of a given configuration $q(1)$ cannot be approximated up to a given absolute/relative error $\varepsilon > 0$ unless $P=NP$, even if all outgoing edges of all stochastic control states in the underlying OC-MDP have probability $1/2$.*

The proof of Theorem 10 is split into two phases, which are relatively independent. First, we show that given a propositional formula φ , one can efficiently compute an OC-MDP \mathcal{A} , a configuration $p(K)$ of \mathcal{A} , and a number N such that the value of $p(K)$ is either $N - 1$ or N depending on whether φ is satisfiable or not, respectively. The numbers K and N are exponential in $\|\varphi\|$, which means that their encoding size is polynomial (we represent all numerical constants in binary). Here we use the technique of encoding propositional assignments into counter values presented in [19], but we also need to invent some specific gadgets to deal with our specific objective. The first part already implies that approximating $\text{Val}(q(i))$ is computationally hard. In the second phase, we show that the same holds also for configurations where the counter is initiated to 1. This is achieved by employing another gadget which just increases the counter to an exponentially high value with a sufficiently large probability. The two phases are elaborated in Lemma 26 and Lemma 29 which can be found in Appendix A.4.

References

1. Proceedings of FST&TCS 2010, Leibniz International Proceedings in Informatics, vol. 8. Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2010)

2. Bach, E., Shallit, J.: Algorithmic Number Theory. Vol. 1, Efficient Algorithms. The MIT Press (1996)
3. Brázdil, T., Brožek, V., Etessami, K.: One-counter stochastic games. In: Proceedings of FST&TCS 2010 [1], pp. 108–119
4. Brázdil, T., Brožek, V., Etessami, K., Kučera, A.: Approximating the termination value of one-counter MDPs and stochastic games. In: Proceedings of ICALP 2011, Part II. Lecture Notes in Computer Science, vol. 6756, pp. 332–343. Springer (2011)
5. Brázdil, T., Brožek, V., Etessami, K., Kučera, A., Wojtczak, D.: One-counter Markov decision processes. In: Proceedings of SODA 2010. pp. 863–874. SIAM (2010)
6. Brázdil, T., Brožek, V., Forejt, V., Kučera, A.: Reachability in recursive Markov decision processes. *Information and Computation* 206(5), 520–537 (2008)
7. Brázdil, T., Brožek, V., Kučera, A., Obdržálek, J.: Qualitative reachability in stochastic BPA games. *Information and Computation* 208(7), 772–796 (2010)
8. Brázdil, T., Kiefer, S., Kučera, A.: Efficient analysis of probabilistic programs with an unbounded counter. In: Proceedings of CAV 2011. Lecture Notes in Computer Science, vol. 6806, pp. 208–224. Springer (2011)
9. Chatterjee, K., Doyen, L.: Energy parity games. In: Proceedings of ICALP 2010, Part II. Lecture Notes in Computer Science, vol. 6199, pp. 599–610. Springer (2010)
10. Chatterjee, K., Doyen, L., Henzinger, T., Raskin, J.F.: Generalized mean-payoff and energy games. In: Proceedings of FST&TCS 2010 [1], pp. 505–516
11. Etessami, K., Wojtczak, D., Yannakakis, M.: Recursive stochastic games with positive rewards. In: Proceedings of ICALP 2008, Part I. Lecture Notes in Computer Science, vol. 5125, pp. 711–723. Springer (2008)
12. Etessami, K., Wojtczak, D., Yannakakis, M.: Quasi-birth-death processes, tree-like QBDs, probabilistic 1-counter automata, and pushdown systems. *Performance Evaluation* 67(9), 837–857 (2010)
13. Etessami, K., Yannakakis, M.: Recursive Markov decision processes and recursive stochastic games. In: Proceedings of ICALP 2005. Lecture Notes in Computer Science, vol. 3580, pp. 891–903. Springer (2005)
14. Etessami, K., Yannakakis, M.: Efficient qualitative analysis of classes of recursive Markov decision processes and simple stochastic games. In: Proceedings of STACS 2006. Lecture Notes in Computer Science, vol. 3884, pp. 634–645. Springer (2006)
15. Filar, J., Vrieze, K.: *Competitive Markov Decision Processes*. Springer (1996)
16. Göller, S., Lohrey, M.: Branching-time model checking of one-counter processes. In: Proceedings of STACS 2010. Leibniz International Proceedings in Informatics, vol. 5, pp. 405–416. Schloss Dagstuhl–Leibniz-Zentrum für Informatik (2010)
17. Jančar, P., Kučera, A., Moller, F., Sawa, Z.: DP lower bounds for equivalence-checking and model-checking of one-counter automata. *Information and Computation* 188(1), 1–19 (2004)
18. Jančar, P., Sawa, Z.: A note on emptiness for alternating finite automata with a one-letter alphabet. *Information Processing Letters* 104(5), 164–167 (2007)
19. Kučera, A.: The complexity of bisimilarity-checking for one-counter processes. *Theoretical Computer Science* 304(1–3), 157–183 (2003)
20. Latouche, G., Ramaswami, V.: *Introduction to Matrix Analytic Methods in Stochastic Modeling*. ASA-SIAM series on statistics and applied probability (1999)
21. Puterman, M.: *Markov Decision Processes*. Wiley (1994)
22. Serre, O.: Parity games played on transition graphs of one-counter processes. In: Proceedings of FoSSaCS 2006. Lecture Notes in Computer Science, vol. 3921, pp. 337–351. Springer (2006)
23. Williams, D.: *Probability with Martingales*. Cambridge University Press (1991)

A Appendix

First, let us fix some additional notation that will be used throughout the whole appendix.

Given a random variable X we denote $\mathbb{E}_{q(i)}^\pi X$ the expected value of X computed under strategy π from initial configuration $q(i)$. Since the initial configuration will be fixed in most of the proofs, we will usually omit the subscript and write only $\mathbb{E}^\pi X$.

Also, given a random variable X and an event A , we use $\mathbb{E}(X \mid A)$ to denote the conditional expectation of X given the event A .

We also use $p_{\min}^\mathcal{A}$ to denote the minimal positive transition probability in \mathcal{A} . We will usually omit the superscript \mathcal{A} if \mathcal{A} is clear from the context.

We say that (finite or infinite) path u in OC-MDP \mathcal{A} *hits* or *reaches* a set $D \subseteq Q$ if $S^{(i)}(u) \in D$ for some $i \leq \text{len}(u)$. We say that u *evades* D if it does not hit D .

A.1 Proof of part (B.2) of Proposition 5.

First, denote \mathcal{A}^σ the finite one-counter Markov chain that results from application of counterless strategy σ on \mathcal{A} . That is, $\mathcal{A}^\sigma = (Q, (Q, \emptyset), \delta, P^\sigma)$ where $P_q^\sigma(q, i, r) = P(q, i, r)$ for every stochastic state q of \mathcal{A} , while for every non-deterministic state q of \mathcal{A} we have that P_q^σ is a Dirac distribution that gives probability 1 to transition selected by $\sigma(q)$.

Theorem 8.8.6 of [21] now guarantees that the set D is exactly the set of all recurrent states in \mathcal{A}^σ . In particular, in \mathcal{A}^σ there is no transition leaving D .

Now, let us recall a fundamental result from theory of linear programming, the Complementary slackness theorem. In essence, this theorem states that whenever we have a pair of solutions u and v of the primal and dual linear program, respectively, then the following equivalence holds: The j -th component of v is positive iff u satisfies the j -th inequality of the primal linear program as an equality. We can apply this on our pair of solutions $(\bar{x}, (\bar{z}_q)_{q \in Q})$, $((\bar{y}_q)_{q \in Q_0}, (\bar{y}_{(q,i,q')})_{q \in Q_1, (q,i,q') \in \delta})$ to obtain the following system of linear equations:

$$\begin{aligned} \bar{z}_q &= -\bar{x} + k + \bar{z}_r && \text{whenever } q \in Q_1 \cap D \text{ and } \sigma \text{ selects } (q, k, r), \\ \bar{z}_q &= -\bar{x} + \sum_{(q,k,r) \in \delta} P_q((q, k, r)) \cdot (k + \bar{z}_r) && \text{for all } q \in Q_0 \cap D. \end{aligned}$$

With the help of these equations, we can easily prove the following lemma:

Lemma 11. *Under strategy σ , for any initial configuration $q(i)$ with $q \in D$ and for any history u of length i we have $\mathbb{E}^\sigma(m^{(i+1)} \mid \text{Run}(u)) = m^{(i)}(u)$. That is, $\{m^{(i)}\}_{i \geq 0}$ is a martingale relative to the filtration $\{\mathcal{F}_i\}_{i \geq 0}$.*

Proof. The proof is the same as the proof of Lemma 22 in [8]. □

From results of [8] (where termination time of one-counter Markov chains was studied) it follows that under strategy σ the expected termination time is finite from every initial configuration of the form $q(i)$ with $q \in D$. To be more specific, we can prove the following:

Lemma 12. *For every initial configuration $q(i)$ with $q \in D$ there are numbers $N \in \mathbb{N}$, $0 < a < 1$ such that for every $n \geq N$ we have $\mathbb{P}_{q(i)}^\sigma(T = n) \leq a^n$.*

Proof. The proof is the same as proof of Proposition 7 in [8]. \square

The finiteness of termination time easily follows because

$$\mathbb{E}^\sigma q(i) = \sum_{k \in \mathbb{N}} k \cdot \mathbb{P}_{q(i)}^\sigma(T = k) \leq N + \sum_{k \geq N} k \cdot \mathbb{P}_{q(i)}^\sigma(T = k) \leq N + \sum_{k \in \mathbb{N}} k \cdot a^k < \infty.$$

Corollary 13. *For any initial configuration $q(i)$ with $q \in D$ we have $\mathbb{E}^\sigma q(i) < \infty$.*

Lemma 14. *For any initial configuration $q(i)$ where $q \in D$ we have $\mathbb{E}^\sigma q(i) \leq (i + V)/|\bar{x}|$.*

Proof. As in proof of part (B.1) of Proposition 5 we want to use the Optional stopping theorem to prove that $\mathbb{E}^\sigma m^{(0)} = \mathbb{E}^\sigma m^{(T)}$. We just need to verify that the assumptions of this theorem hold. We have already argued that $\{m^{(i)}\}_{i \geq 0}$ is a martingale and T is a stopping time relative to the same filtration $\{\mathcal{F}_i\}_{i \geq 0}$. We have also observed that $\{m^{(i)}\}_{i \geq 0}$ has bounded differences. From the previous corollary we also now, that the expectation of stopping time T is finite. Thus, the Optional stopping theorem applies and we indeed have $\mathbb{E}^\sigma m^{(0)} = \mathbb{E}^\sigma m^{(T)}$. But $\mathbb{E}^\sigma m^{(0)} = i + \bar{z}_q$ and $\mathbb{E}^\sigma m^{(T)} = \mathbb{E}^\sigma \bar{z}_{S(T)} + |\bar{x}| \cdot \mathbb{E}^\sigma q(i)$. This gives us $\mathbb{E}^\sigma q(i) = (i + \bar{z}_q - \mathbb{E}^\sigma \bar{z}_{S(T)})/|\bar{x}| \leq (i + V)/|\bar{x}|$. \square

To prove part (B.2) of Proposition 5 it remains to prove the upper bound for arbitrary initial state. Intuitively, every state outside D is transient in \mathcal{A}^σ and thus under σ we must reach D “quickly”. Once D is reached, we can apply the bound from previous lemma.

Lemma 15. *Let $q(i)$ be any initial configuration. Denote $p := \exp(-p_{\min}^{|Q|}/|Q|)$ where p_{\min} is the minimal nonzero probability in \mathcal{A} . Then we have*

$$\mathbb{E}^\sigma q(i) \leq \frac{i + V + 2|Q| + \frac{4}{(1-p)^2}}{|\bar{x}|}.$$

Before we prove Lemma 15, we should mention that the Lemma directly implies inequality in part (B.2) of Proposition 5. Indeed, the desired inequality holds for $U = V + 2|Q| + \frac{4}{(1-p)^2}$. The required asymptotic bound on U is easy to check: we just need to recall, that for every real number $x \in [0, 1]$ we have $1 - \exp(-x) \geq x/2$ and thus $1/(1-p)^2 \leq 4|Q|^2/p_{\min}^{2|Q|}$. This also shows that U is computable in time polynomial in $\|\mathcal{A}\|$.

Proof (Proof of Lemma 15). We can write

$$\mathbb{E}^\sigma q(i) = \mathbb{E}^\sigma(T_1 + T_2),$$

where $T_1(\omega) = k$ iff k is the first point in time when either $C^{(k)}(\omega) = 0$ or $S^{(k)}(\omega) \in D$; and where T_2 returns the termination time measured from the first time when D was hit (formally we have $T_2(\omega) = -T_1(\omega) + T(\omega)$ if $T_1(\omega) < \infty$ and $T_2(\omega) = 0$ otherwise). We will bound expectations of T_1 and T_2 separately.

Let's start with T_1 . Any run ω with $T_1(\omega) \geq k$ must either terminate before hitting D but after at least k steps; or it has to hit D after at least k steps. In both cases ω has to evade D for at least $k - 1$ steps. From e.g. Lemma 23 of [8] we know, that probability of evading D for at least $k - 1 \geq |Q| - 1$ steps is at most $2p^k$. We get

$$\begin{aligned} \mathbb{E}^\sigma T_1 &= \sum_{k=1}^{\infty} k \cdot \mathbb{P}_{q(i)}^\sigma(T_1 = k) = \sum_{k=1}^{\infty} \mathbb{P}_{q(i)}^\sigma(T_1 \geq k) \leq |Q| + \sum_{k=|Q|+1}^{\infty} \mathbb{P}_{q(i)}^\sigma(T_1 \geq k) \\ &\leq |Q| + \sum_{k=|Q|+1}^{\infty} 2p^k \leq |Q| + \sum_{k=0}^{\infty} 2p^k \leq |Q| + \frac{2}{1-p}. \end{aligned} \quad (1)$$

Let us now concentrate on T_2 . For any $l > 0$ we denote D_l the set of all runs that terminate after hitting D and have a counter value l when they hit D for the first time. (Formally, $\omega \in D_l$ iff $T_1(\omega) < \infty$, $S^{(T_1)}(\omega) \in D$ and $C^{(T_1)}(\omega) = l$.) We also denote D_0 the set of all runs that reach a configuration with zero counter before or simultaneously with hitting D for the first time. Then we have

$$\mathbb{E}^\sigma T_2 = \sum_{l=0}^{\infty} \mathbb{E}^\sigma(T_2 \mid D_l) \cdot \mathbb{P}_{q(i)}^\sigma(D_l). \quad (2)$$

Note that by Lemma 14 we have for every $l \in \mathbb{N}$

$$\mathbb{E}^\sigma(T_2 \mid D_l) \leq \frac{l + V}{|\vec{x}|}$$

Particularly for every $l \leq i + |Q|$ we have

$$\mathbb{E}^\sigma(T_2 \mid D_l) \leq \frac{i + V}{|\vec{x}|} + \frac{|Q|}{|\vec{x}|}. \quad (3)$$

On the other hand, for $l \geq i + |Q|$ we have $\mathbb{P}_{q(i)}^\sigma(D_l) \leq 2p^{l-i}$, since no run in D_l can hit D in less than $l - i$ steps. Moreover, for $l \geq i + |Q|$ we can write

$$\mathbb{E}^\sigma(T_2 \mid D_l) \leq \frac{i + (l - i) + V}{|\vec{x}|} = \frac{i + V}{|\vec{x}|} + \frac{(l - i)}{|\vec{x}|}. \quad (4)$$

Plugging (3) and (4) into (2) we can compute

$$\mathbb{E}^\sigma T_2 \leq \frac{i + V}{|\vec{x}|} + \frac{|Q|}{|\vec{x}|} + \sum_{l \geq i + |Q|} \frac{2 \cdot (l - i) \cdot p^{l-i}}{|\vec{x}|} \leq \frac{i + V}{|\vec{x}|} + \frac{|Q|}{|\vec{x}|} + \frac{2}{|\vec{x}| \cdot (1 - p)^2}. \quad (5)$$

Putting (1) and (5) together we obtain

$$\mathbb{E}^\sigma \leq \frac{i + V}{|\vec{x}|} + \frac{|Q|}{|\vec{x}|} + \frac{4}{|\vec{x}| \cdot (1 - p)^2} + |Q| \leq \frac{i + V + 2|Q| + \frac{4}{(1-p)^2}}{|\vec{x}|}.$$

□

A.2 Proof of Proposition 8

First, consider the membership problem for $Q_{<0}$. A decomposition of Q into maximal end components can be computed in polynomial time using standard algorithms (see, e.g. [21]). By solving the system \mathcal{L} for individual MECs, we obtain the trends \bar{x}_C that in turn determine the set H . Finally, solving, in polynomial time, the qualitative reachability of H for every state q we obtain the set $Q_{<0}$.

It remains to prove that $Q_{fin} = Q_{<0}$. We prove both inclusions separately.

‘ \supseteq ’: Assume that $p \in Q_{<0}$. First, observe that if p belongs to a MEC C satisfying $\bar{x}_C < 0$ then, by Proposition 5, there is a counterless strategy which stays in C and terminates in finite expected time. In particular, $\text{Val}(p(\ell))$ is finite and depends linearly on ℓ .

Assume that a strategy σ almost surely reaches H from p in \mathcal{A}_M . As almost sure reachability is solved using memory-less strategies in finite MDPs, we may assume that σ is memory-less. Denote by \mathcal{H} the set of all configuration of the form $q(\ell)$ where either $q \in H$, or $\ell = 0$. The strategy σ induces a counter-less strategy σ' in \mathcal{A}_M^∞ which reaches \mathcal{H} with probability one. Moreover, using σ' , \mathcal{H} is reachable with a positive probability from any configuration in at most $|Q|$ steps. This means that the expected time to reach \mathcal{H} is finite and the probability of reaching a configuration of \mathcal{H} with counter value at most ℓ before any other configuration of \mathcal{H} is bounded by $\frac{c}{d^\ell}$ for suitable constants $c, d > 0$. As $\text{Val}(q(\ell))$ depends linearly on ℓ for every $q \in H$, we obtain that the expected termination time for $p(k)$ is finite.

‘ \subseteq ’: We proceed by contradiction. Assume that $Q_{fin} \setminus Q_{<0} \neq \emptyset$. The following Lemma formalizes the crucial idea.

Lemma 16. *Assuming $Q_{fin} \setminus Q_{<0} \neq \emptyset$, there is a MEC C satisfying $C \subseteq Q_{fin} \setminus Q_{<0}$ for which the following holds: if $s \rightsquigarrow t$ where $s \in C$ and $t \notin C$, then $t \in Q \setminus Q_{fin}$.*

Proof. First, we prove that if $Q_{fin} \setminus Q_{<0} \neq \emptyset$, then it contains at least one MEC. Assume, to the contrary, that all MECs contained in Q_{fin} are also contained in $Q_{<0}$. We claim that then $Q_{fin} \subseteq Q_{<0}$. Indeed, consider $p \in Q_{fin} \setminus Q_{<0}$. Note that starting in p , almost every run eventually reaches a MEC no matter what strategy is used. Moreover, there is a strategy which almost surely stays within Q_{fin} forever starting in p . Using such a strategy, almost all runs initiated in p reach MECs contained in Q_{fin} and hence also in $Q_{<0}$. Thus, by definition of $Q_{<0}$, we have $p \in Q_{<0}$ which contradicts $p \in Q_{fin} \setminus Q_{<0}$.

If there is a MEC $C \subseteq Q_{fin} \setminus Q_{<0}$ such that no transition $s \rightsquigarrow t$ satisfies $s \in C$ and $t \notin C$, then we are done. Assume, to obtain a contradiction, that for every MEC $C \subseteq Q_{fin} \setminus Q_{<0}$ there is $s_C \rightsquigarrow t_C$ such that $s_C \in C$ but $t_C \in Q_{fin} \setminus C$. Then for every t_C there is a strategy which stays within Q_{fin} . Let us consider a strategy π that does the following:

- in all states of every MEC C satisfying $C \subseteq Q_{fin} \setminus Q_{<0}$, the strategy π strives to reach s_C with probability one
- in each s_C , the strategy π takes the transition $s_C \rightsquigarrow t_C$ with probability one
- in states of Q_{fin} that do not belong to any MEC, the strategy π stays in Q_{fin} .

Note that we may safely assume that π is memory-less. Consider the Markov chain M^π induced by π on states of Q_{fin} . There are two possibilities. First, every bottom strongly connected component (BSCC) of M^π contains a state of $Q_{<0}$. Then $Q_{<0}$ is reachable

with probability one using π from states of $Q_{fin} \setminus Q_{<0}$, a contradiction with definition of $Q_{<0}$. Assume that there is at least one BSCC of M_π which does not contain states of $Q_{<0}$. However, then the BSCC contains only states of $Q_{fin} \setminus Q_{<0}$. Thus, by definition of π , the BSCC must contain at least two MECs, a contradiction with the definition of MEC. \square

Now let ℓ be a counter value such that for every $q \in Q \setminus Q_{fin}$ we have that $\text{Val}(q(\ell)) = \infty$. Let σ be a strategy and consider $p(\ell + |Q|)$ where $p \in C$. We prove that p cannot belong to Q_{fin} which contradicts $C \subseteq Q_{fin} \setminus Q_{<0}$.

There are two cases. First, assume that using σ , a configuration of the form $q(k)$, where $k \geq \ell$ and $q \in Q \setminus C$, is reachable via configurations with counter values at least ℓ whose control states belong to C . Then by Lemma 16, $q \in Q \setminus Q_{fin}$ and thus the expected termination time from $q(\ell)$ is infinite. It follows that the termination time from $p(\ell + |Q|)$ using σ is infinite as well. Assume that there is no such a path, i.e. that the only way how to leave C from $p(\ell + |Q|)$ using σ is to decrease the counter value below ℓ . But then the expected termination time from $p(\ell + |Q|)$ using σ is at least as large as $\text{Val}(p(|Q|))$ in \mathcal{A}_C , which is infinite by Proposition 5 due to $\bar{x}_C \geq 0$. In both cases we obtain that $p \notin Q_{fin}$, a contradiction.

Note that the number l mentioned above can be bounded from above by $|Q|$ by Lemma 7.

A.3 Proof of Proposition 9

First we introduce some notation: for any run ω we denote $\text{inf}(\omega)$ the set of states that are visited infinitely often by ω . For any MEC C we denote $M_C = \{\omega \mid \text{inf}(\omega) \subseteq C\}$. It is well known that under arbitrary strategy π we have $\mathbb{P}^\pi\left(\bigcup_{C \in \text{MEC}(\mathcal{A})} M_C\right) = 1$, i.e. that $\text{inf}(\omega)$ is almost surely contained in some MEC.

For any state q denote $\Sigma_q^{<0}$ the set of all strategies σ with the property that $\mathbb{P}_q^\sigma(\{\omega \mid \omega \in M_C, \bar{x}_C \geq 0\}) = 0$. Note that by Proposition 8 we have $\Sigma_q^{<0} \neq \emptyset$ for all $q \in Q_{fin}$.

Let us start with part (B) of Proposition 9. We want to describe a counterless strategy σ that terminates “quickly” from any configuration $q(j)$ with $q \in Q_{fin}$. Part (B2) of Proposition 5 gives us for every MEC C counterless strategy σ_C such that for any initial configuration $q(i)$ with $q \in C$ we have $\mathbb{E}^{\sigma_C} q(i) \leq (i + U_C)/|\bar{x}_C|$, for some number U_C . Main idea behind construction of σ is to stitch these strategies together in appropriate way.

We argue that the following should hold: First, strategy σ should be in $\Sigma_q^{<0}$ for all states $q \in Q_{fin}$. Otherwise, the finite Markov chain \mathcal{A}^σ induced by σ on states of \mathcal{A} would have some bottom strongly connected component (BSCC) contained in MEC C with $\bar{x}_C \geq 0$. By part (A) of Proposition 5 this would mean that $\mathbb{E}^\sigma q(j) = \infty$ for some j .

Second, strategy σ should minimize the long-run average number of steps needed to decrease the counter value by one. Note that since \bar{x}_C represents the minimal long-run average change in counter value in MEC C , the number $|\bar{x}_C|^{-1}$ represents exactly the long-run average time needed to decrease the counter by 1 in C , provided that $\bar{x}_C < 0$. Thus, strategy σ should minimize the weighted sum $\sum_{C \in \text{MEC}(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(M_C) \cdot |\bar{x}_C|^{-1}$ for any initial configuration $q(i)$ with $q \in Q_{fin}$. Note that the objective of minimizing

$\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(M_C) \cdot |\bar{x}_C|^{-1}$ does not depend in any way on counter values so it suffices to show that the sum is minimized for some (unspecified) initial counter value i .

More formally, for every state $q \in Q_{fin}$ there is unique number $t_q < 0$ such that $|t_q|^{-1} = \inf_{\pi \in \Sigma_q^{<0}} \sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\pi(M_C) \cdot |\bar{x}_C|^{-1}$ for all i . We call t_q the *minimal trend* achievable from q . Our goal is to find counterless deterministic strategy $\sigma \in \Sigma_q^{<0}$ such that $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(M_C) \cdot |\bar{x}_C|^{-1} = |t_q|^{-1}$, for every $q \in Q_{fin}$ and every i .

Denote $\bar{x}_0 = \max\{\bar{x}_C \mid C \in MEC(\mathcal{A}), \bar{x}_C < 0\}$. In order to compute strategy σ and numbers t_q , we transform \mathcal{A} into a new finite-state MDP with rewards \mathcal{A}_R by “forgetting” counter changes in \mathcal{A} and defining a reward function R on transitions in \mathcal{A} as follows:

$$R(s \leadsto t) = \begin{cases} \frac{1}{\bar{x}_C} & \text{if } s, t \in C \text{ and } \bar{x}_C < 0 \\ \frac{\bar{x}_0^{-1} - 1}{p_{\min}^{|\mathcal{Q}|}} & \text{otherwise,} \end{cases}$$

It is clear that \mathcal{A}_R can be constructed in time polynomial in $\|\mathcal{A}\|$.

Claim. In \mathcal{A}_R the maximal average reward achievable from state $q \in Q_{fin}$ is equal to t_q^{-1} . Moreover, there is a memoryless deterministic strategy σ_R in \mathcal{A}_R such that for every state $q \in Q_{fin}$ we have $\sigma_R \in \Sigma_q^{<0}$ and $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_q^{\sigma_R}(M_C) \cdot |\bar{x}_C|^{-1} = |t_q|^{-1}$.

Proof. The existence of optimal memoryless deterministic strategy σ for maximization of average reward follows from standard results on MDPs (see [21]). It is obvious that for any $q \in Q_{fin}$ and any strategy $\pi \in \Sigma_q^{<0}$ the average reward obtained with strategy π in \mathcal{A}_R is equal to $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_q^\pi(M_C) \cdot |\bar{x}_C|^{-1}$. It thus suffices to prove that $\sigma_R \in \Sigma_q^{<0}$ for every $q \in Q_{fin}$. Denote M^{σ_R} the finite Markov chain induced by σ_R on states of \mathcal{A} . Assume, for the sake of contradiction, that $\sigma_R \notin \Sigma_q^{<0}$ for some $q \in Q_{fin}$. Then there must be a BSCC B of M^{σ_R} reachable from q that is contained in some MEC C with $\bar{x}_C \geq 0$. In M_{σ_R} there must be a path of length at most $|\mathcal{Q}|$ from q to B , which means that under σ_R the probability of runs that have average reward $\frac{\bar{x}_0^{-1} - 1}{p_{\min}^{|\mathcal{Q}|}}$ is at least $p_{\min}^{|\mathcal{Q}|}$. Since no run in \mathcal{A}_R has average reward greater than -1 , it follows that average reward achieved from q with σ_R is at most $\bar{x}_0^{-1} - 1 - (1 - p_{\min}^{|\mathcal{Q}|}) < \bar{x}_0^{-1}$. But this is contradiction with σ_R maximizing the average reward, since $\Sigma_q^{<0} \neq \emptyset$ and every strategy from $\Sigma_q^{<0}$ yields average reward at least \bar{x}_0^{-1} . \square

Strategy σ_R can be computed in polynomial time with standard algorithms (see, e.g., [21]). We can now construct the desired counterless strategy σ as follows: denote \mathcal{A}^{σ_R} the finite Markov chain induced by σ_R on states of \mathcal{A} . Note that every bottom strongly connected component of \mathcal{A}^{σ_R} is contained in exactly one MEC $C(B)$ of \mathcal{A} . Strategy σ behaves in the same way as σ_R until some BSCC B of \mathcal{A}^{σ_R} is reached. Then σ starts to behave as $\sigma_{C(B)}$. It is easy to see that $\sigma \in \Sigma_q^{<0}$ for all states $q \in Q_{fin}$.

Clearly, for every MEC C and every initial configuration $q(i)$ with $q \in Q_{fin}$ we have $\mathbb{P}_q^{\sigma_R}(M_C) = \mathbb{P}_{q(i)}^{\sigma_R}(M_C)$ and thus also $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(M_C) \cdot |\bar{x}_C|^{-1} = |t_q|^{-1}$ for every i . Note that numbers t_q satisfy all conditions mentioned in the initial part of Proposition 9. Moreover, we can prove the following upper bound on expected termination time under σ :

Proposition 17. *There is a number $U \in \exp(\|\mathcal{A}\|^{O(1)})$ that is computable in time polynomial in $\|\mathcal{A}\|$ such that for any initial configuration $q(i)$ with $q \in Q_{fin}$ and $i \geq |Q|$ we have*

$$\mathbb{E}^\sigma q(i) \leq \frac{i}{|t_q|} + U.$$

Proof. The proof closely follows the proof of Lemma 15. However, there is a new obstacle in a presence of components with different trends.

Since the strategy σ is memoryless, its application on \mathcal{A} yields a finite one-counter Markov chain \mathcal{A}^σ . Denote D the union of its bottom strongly connected components. We can now write

$$\mathbb{E}^\sigma q(i) = \mathbb{E}^\sigma (T_1 + T_2), \quad (6)$$

where again $T_1(\omega) = k$ iff k is the first point in time when ω hits either D or reaches a configuration with a zero counter and T_2 is a time to hit a configuration with a zero counter after hitting D (T_2 returns zero if the run never terminates or terminates before hitting D).

We will bound expectations of T_1 and T_2 separately.

The bound on $\mathbb{E}^\sigma T_1$ can be computed in exactly the same way as in Lemma 15. Thus we can conclude that

$$\mathbb{E}^\sigma T_1 \leq |Q| + \frac{2}{1-p}, \quad (7)$$

where $p = \exp(-p_{\min}^{|Q|}/|Q|)$.

Now we bound the expectations of T_2 . Recall that for any $l > 0$ we denote D_l the set of all runs that terminate after reaching D and have a value counter value exactly l when they hit D for the first time (and we denote D_0 the set of all runs that terminate before hitting D or hit D with counter value exactly 0). Also recall, that M_C denotes the set of all runs ω with $\inf(\omega) \subseteq C$ and that under arbitrary strategy π we have $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}^\pi(M_C) = 1$. Finally, denote $D_l^C = M_C \cap D_l$.

As discussed in section 3.2, we can apply Proposition 5 to every MEC C of \mathcal{A} separately. Especially, by construction of σ the following holds: for every MEC C that contains some BSCC of \mathcal{A}^σ , the Proposition 5 gives us number $U_C \in \exp(\|\mathcal{A}\|^{O(1)})$ such that $\mathbb{E}^\sigma p(j) \leq (j + U_C)/|\bar{x}_C|$, for every $p \in C$ and $j \geq 0$. Set

$$U' = \max\{U_C \mid C \in MEC(\mathcal{A}), C \text{ contains some BSCC of } \mathcal{A}^\sigma\}.$$

Clearly we still have $U' \in \exp(\|\mathcal{A}\|^{O(1)})$.

We have

$$\mathbb{E}^\sigma T_2 = \sum_{C \in MEC(\mathcal{A})} \sum_{l=0}^{\infty} \mathbb{E}^\sigma (T_2 \mid D_l^C) \cdot \mathbb{P}_{q(i)}^\sigma(D_l^C). \quad (8)$$

As in proof of Lemma 15, we can easily show that for any MEC C and any $l \leq i + |Q|$ we have

$$\mathbb{E}^\sigma (T_2 \mid D_l^C) \leq \frac{i + |Q| + U'}{|\bar{x}_C|}. \quad (9)$$

For every C and every $l \geq i + |Q|$ we have

$$\mathbb{E}^\sigma (T_2 \mid D_l^C) \leq \frac{i + U'}{|\bar{x}_C|} + \frac{(l - i)}{|\bar{x}_C|} \quad (10)$$

and $\mathbb{P}_{q(j)}^\sigma(D_l) \leq 2p^{l-i}$ (the latter holds by Lemma 23 of [8]).

Recall that we have denoted $\bar{x}_0 = \max\{\bar{x}_C \mid C \in MEC(\mathcal{A}), \bar{x}_C < 0\}$. Putting (9) and (10) together we obtain for any fixed $C \in MEC(\mathcal{A})$

$$\begin{aligned} \sum_{l=0}^{\infty} \mathbb{E}^\sigma(T_2 \mid D_l^C) \cdot \mathbb{P}_{q(i)}^\sigma(D_l^C) &\leq \frac{i + |Q| + U'}{|\bar{x}_C|} \cdot \mathbb{P}_{q(i)}^\sigma(M_C) + \sum_{l=i+|Q|}^{\infty} \frac{(l-i) \cdot \mathbb{P}_{q(i)}^\sigma(D_l^C)}{|\bar{x}_C|} \\ &\leq \frac{i + |Q| + U'}{|\bar{x}_C|} \cdot \mathbb{P}_{q(i)}^\sigma(M_C) + \sum_{l=i+|Q|}^{\infty} \frac{(l-i) \cdot \mathbb{P}_{q(i)}^\sigma(D_l^C)}{|\bar{x}_0|}. \end{aligned}$$

Moreover, from the definition of strategy σ we know that $\sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(M_C) \cdot |\bar{x}_C|^{-1} = |t_q|^{-1}$. We can use this and continue from (8) as follows:

$$\begin{aligned} \mathbb{E}^\sigma T_2 &\leq \sum_{C \in MEC(\mathcal{A})} \left(\frac{i + |Q| + U'}{|\bar{x}_C|} \cdot \mathbb{P}_{q(i)}^\sigma(M_C) + \sum_{l=i+|Q|}^{\infty} \frac{(l-i) \cdot \mathbb{P}_{q(i)}^\sigma(D_l^C)}{|\bar{x}_0|} \right) \\ &= \sum_{C \in MEC(\mathcal{A})} \left(\frac{i + |Q| + U'}{|\bar{x}_C|} \cdot \mathbb{P}_{q(i)}^\sigma(M_C) \right) + \sum_{l=0}^{\infty} \frac{(l-i) \cdot \sum_{C \in MEC(\mathcal{A})} \mathbb{P}_{q(i)}^\sigma(D_l^C)}{|\bar{x}_0|} \\ &= \frac{i + |Q| + U'}{|t_q|} + \sum_{l=0}^{\infty} \frac{\overbrace{(l-i) \cdot \mathbb{P}_{q(i)}^\sigma(D_l)}^{\leq 2p^{l-i}}}{|\bar{x}_0|} \leq \frac{i + |Q| + U'}{|t_q|} + \frac{2}{|\bar{x}_0| \cdot (1-p)^2}. \quad (11) \end{aligned}$$

Combining (7) and (8) we can conclude that

$$\mathbb{E}^\sigma q(i) \leq \frac{i}{|t_q|} + \frac{2|Q| + U'}{|t_q|} + \frac{4}{|\bar{x}_0| \cdot (1-p)^2} \leq \frac{i}{|t_q|} + \frac{2|Q| + U'}{|\bar{x}_0|} + \frac{4}{|\bar{x}_0| \cdot (1-p)^2}.$$

The inequality in Proposition 17 thus holds for $U = \frac{2|Q| + U'}{|\bar{x}_0|} + \frac{4}{|\bar{x}_0| \cdot (1-p)^2}$. The desired asymptotic bound is again easy to check. \square

It remains to prove part (B) of Proposition 9 (with numbers t_q being the minimal trends achievable from q).

The following Claim shows, that in order to prove Proposition 9 (B) it suffices to prove its validity for strategies in $\Sigma_q^{<0}$, because termination value under some arbitrarily fixed strategy can be approximated up to some exponential error by termination value under suitable strategy from $\Sigma_q^{<0}$.

Claim. There is a number $K_1 \in \exp(\|\mathcal{A}\|^{O(1)})$ that is computable in time polynomial in $\|\mathcal{A}\|$, with the following property: for every strategy π and any initial configuration $q(i)$ with $i \geq |Q|$ there is a strategy $\pi' \in \Sigma_q^{<0}$ such that $\mathbb{E}^\pi q(i) \geq \mathbb{E}^{\pi'} q(i) - K_1$.

Proof. Set $K_1 = (|Q| + U)/|\bar{x}_0|$, where U is the constant from Proposition 17. Fix arbitrary strategy π . If $\mathbb{E}^\pi q(i) = \infty$, then the inequality clearly holds for any strategy $\pi' \in \Sigma_q^{<0}$. Otherwise, since $i \geq |Q|$, with π we must almost surely reach a configuration

of the form $p(|Q|)$. For every such reachable configuration we must have $p \in Q_{fin}$, since otherwise we would have $\mathbb{E}^\pi q(i) = \infty$ by Lemma 7. Define new strategy π' as follows: π' behaves in the same way as π until the configuration with counter height $|Q|$ is reached: then it starts to behave as strategy σ from Proposition 9. Then clearly $\pi' \in \Sigma_q^{<0}$ and by Proposition 17 the switch to strategy σ in height $|Q|$ cannot delay the termination for more than $(|Q| + U)/|\bar{x}_0|$ steps. \square

Under strategy $\pi \in \Sigma_q^{<0}$ we never reach state from $Q \setminus Q_{fin}$, if we start from q . We can thus safely remove all states from $Q \setminus Q_{fin}$, together with adjacent transitions, without influencing the behavior under strategies from $\Sigma_q^{<0}$. In the following we can without loss of generality assume that $Q = Q_{fin}$ and that all strategies are in $\Sigma_q^{<0}$, for every state q .

We will now finish the proof in two steps. First, we observe that there is only a small probability that the run revisits (i.e. leaves and then visits again) some MEC many times. Actually, this probability decays exponentially in number of revisits. We call a transition $r(j) \rightsquigarrow r'(j')$ in $\mathcal{M}_{\mathcal{A}}^\infty$ a *switch* if there exists some MEC C such that $\{r, r'\} \cap C = 1$. For any run ω we denote $\sharp(\omega)$ the number of switches on ω and we set $W(\omega) = \sharp(\omega) + 1$. That is, random variable W counts the number of maximal time intervals in which ω either stays within a single MEC or outside any MEC.

Lemma 18. *For every strategy π , every initial configuration $q(i)$ and every $k \in \mathbb{N}$*

$$\mathbb{P}_{q(i)}^\pi(W = k) \leq 8 \cdot |Q| \cdot c^k,$$

where $c = \exp\left(\frac{-p_{\min}^{|Q|}}{2|Q|}\right)$.

Proof. If $p_{\min} = 1$, i.e. there are no (truly) stochastic states, then MECs are actually strongly connected components, $W(\omega) \leq 2 \cdot |Q|$ for every run ω , and the Lemma trivially holds. Otherwise, we have $p_{\min} \leq 1/2$. We can use the following:

Claim. Let \mathcal{A} be arbitrary OC-MDP and let C be a MEC of \mathcal{A} . Further, let $q \notin C$ be any state that can be reached from C with probability 1 (under some strategy). Then, under arbitrary strategy, the probability of reaching C from any initial configuration of the form $q(i)$ is at most $1 - p_{\min}^{|Q|}$.

Let ρ be the strategy that maximizes the probability of reaching C from q in \mathcal{A}_M . From standard results on MDPs we may assume that ρ is memoryless. Denote M^ρ the finite Markov chain induced by ρ on states of \mathcal{A}_M . There must be at least one BSCC B of M^ρ reachable from q such that in \mathcal{A}_M the probability of reaching C from any state of B is less than 1 under any strategy (otherwise, there would be a strategy that almost surely reaches C from q – a contradiction with C being a MEC). In particular, sets B and C are disjoint. Thus, the probability of *not reaching* C from q under ρ is at least as large as probability of hitting B in M^ρ . Since ρ is memoryless, there is a run in M^ρ that reaches B in at most $|Q|$ steps. Thus, the probability of hitting B is at least $p_{\min}^{|Q|}$.

Let us now finish proof of the Lemma. For any MEC C and any $l \in \mathbb{N}$ denote R_C^l the set of all runs that leave a MEC C and then return to it for at least l times. The claim

shows that under any strategy π we have $\mathbb{P}_{q(i)}^\pi(R_C^l) \leq (1 - p_{\min}^{|Q|})^l$. Now if $W(\omega) = k$ then ω must have revisited some MEC C at least $\lfloor \frac{k}{2|Q|} - 2 \rfloor$ times, i.e. $\omega \in R_C^{\lfloor \frac{k}{2|Q|} - 2 \rfloor}$ for some MEC C . Thus $\mathbb{P}_{q(i)}^\pi(W = k) \leq |Q| \cdot (1 - p_{\min}^{|Q|})^{\lfloor \frac{k}{2|Q|} - 2 \rfloor}$. Denote $\alpha = (1 - p_{\min}^{|Q|})$.

We have $\lfloor \frac{k}{2|Q|} - 2 \rfloor \leq \frac{k}{2|Q|} - 3$ and thus $\mathbb{P}_{q(i)}^\pi(W = k) \leq |Q| \cdot \alpha^{\frac{k}{2|Q|} - 3} = |Q| \cdot \alpha^{\frac{k}{2|Q|}} / \alpha^3$. Since $p_{\min} \leq 1/2$, we have $1/\alpha^3 \leq 8$. Moreover, from calculus we know that for any real number x we have $1 - x \leq \exp(-x)$. This gives us $\mathbb{P}_{q(i)}^\pi(W = k) \leq 8 \cdot |Q| \cdot \exp\left(-\frac{p_{\min}^{|Q|}}{2|Q|}\right)^k$, and the proof is finished. \square

The crucial idea behind the proof of Proposition 9 (B) is now the following: whenever the system stays either in some MEC or outside any MEC for some period of time, we may approximate its behavior (up to some constant error) using the results of section 3.1 and standard probabilistic computations, respectively. We show, that it is possible to use these approximations to approximate the behavior of the whole system. The error of this new approximation now depends on the average number of time intervals when run stays in some or outside any MEC. The following crucial proposition formalizes this idea.

Proposition 19. *There is a number $K \in \exp\left(\|\mathcal{A}\|^{O(1)}\right)$ that is computable in time polynomial in $\|\mathcal{A}\|$, such that for every memoryless deterministic strategy π and every initial configuration $q(i)$ we have*

$$\mathbb{E}^\pi q(i) \geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W.$$

Before we present the rather technical proof of Proposition 19, let us make sure that it already implies Proposition 9.

Let $q(i)$ be any initial configuration. Fix a memoryless deterministic strategy π that minimizes the expected termination time from $q(i)$. From Lemma 18 we have $\mathbb{E}^\pi W = \sum_{k=0}^{\infty} k \cdot \mathbb{P}_{q(i)}^\pi(W = k) \leq 8 \cdot |Q| \cdot \sum_{k=0}^{\infty} k \cdot c^k = \frac{8 \cdot |Q|}{(1-c)^2}$. From calculus we now that for every $0 \leq x \leq 1$ it holds $1 - \exp(-x) \geq x/2$ and thus we have $E^\pi W \leq \frac{32 \cdot |Q|^2}{p_{\min}^{|Q|}}$. Denote this upper bound K' . Clearly $K' \in \exp\left(\|\mathcal{A}\|^{O(1)}\right)$ is computable in time polynomial in $\|\mathcal{A}\|$. By Proposition 19 we have

$$\text{Val}(q(i)) = \mathbb{E}^\pi q(i) \geq \frac{i}{|t_q|} - K \cdot K'.$$

Since $K \cdot K' \in \exp\left(\|\mathcal{A}\|^{O(1)}\right)$, this proves Proposition 9, which is what we needed to finish the proof of Theorem 3 in general case.

Proof of Proposition 19

Recall, that in the following we assume $Q = Q_{fin}$.

First, we need to present some technical observations.

The following lemma is a slight generalization of part (B1) of Proposition 5.

Lemma 20. *Let $q(l)$ be any initial configuration such that $q \in C$, for some MEC C of \mathcal{A} . Denote T_{\rightarrow} the random variable that returns the first point in time when the run either terminates or reaches configuration of the form $r(j)$ with $r \notin C$. Then under arbitrary deterministic strategy π that satisfies $\mathbb{E}^{\pi} q(l) < \infty$ we have $\mathbb{E}^{\pi} T_{\rightarrow} \geq \frac{l - V_C - 1 - \mathbb{E}^{\pi} C^{(T_{\rightarrow})}}{|\bar{x}_C|}$.*

Proof. Fix an arbitrary initial configuration $q(l)$ and deterministic strategy π with $\mathbb{E}^{\pi} q(l) < \infty$. Consider the following stochastic process $\{\hat{m}^{(i)}\}_{i \geq 0}$:

$$\hat{m}^{(i)} := \begin{cases} C^{(i)} + \bar{z}_{S^{(i)}} - i \cdot \bar{x}_C & \text{if } T_{\rightarrow} \geq i \text{ and } S^{(i)} \in C, \\ C^{(i)} + 1 + \bar{z}_{S^{(i-1)}} - i \cdot \bar{x}_C & \text{if } T_{\rightarrow} \geq i \text{ and } S^{(i)} \notin C, \\ \hat{m}^{(i-1)} & \text{otherwise.} \end{cases}$$

We claim that $\{\hat{m}^{(i)}\}_{i \geq 0}$ is a submartingale relative to the filtration $\{\mathcal{F}_i\}_{i \geq 0}$. The proof is again essentially the same as proof of Lemma 3 in [4]. First, the value of $\hat{m}^{(i)}(\omega)$ clearly depends only on finite prefix of ω of length i . Now let u be any history of length i . If $C^{(j)}(u) = 0$ for some $0 \leq j < i$ or $S^{(j)} \notin C$ for some $0 \leq j \leq i$ (i.e. if $T_{\rightarrow}(\omega) < i$ for all $\omega \in \text{Run}(u)$), then clearly $\mathbb{E}^{\pi}(\hat{m}^{(i+1)} \mid \text{Run}(u)) = \hat{m}^{(i)}(u)$.

Otherwise we denote $r(j)$ the last configuration on u and for every possible successor $r'(j')$ of $r(j)$ in $\mathcal{M}_{\mathcal{A}}^{\infty}$ we set

$$p_{r'(j')} = \begin{cases} \pi(u)(r(j) \rightsquigarrow r'(j')) & \text{if } r \in Q_1 \\ \text{Prob}(r(j))(r(j) \rightsquigarrow r'(j')) & \text{if } r \in Q_0. \end{cases}$$

Suppose that $r \in Q_1$ and that π selects a transition to a configuration $r'(j')$ with $r' \notin C$. Then

$$\begin{aligned} \mathbb{E}^{\pi}(\hat{m}^{(i+1)} \mid \text{Run}(u)) &= \mathbb{E}^{\pi}(C^{(i+1)} + 1 + \bar{z}_{S^{(i)}} - (i+1) \cdot \bar{x}_C \mid \text{Run}(u)) \\ &= C^{(i)}(u) + \underbrace{\mathbb{E}^{\pi}(C^{(i+1)} - C^{(i)} - \bar{x}_C + 1 + \bar{z}_{S^{(i)}} \mid \text{Run}(u))}_{\geq 0} - i \cdot \bar{x}_C \\ &\geq C^{(i)}(u) + \bar{z}_{S^{(i)}(u)} - i \cdot \bar{x}_C = \hat{m}^{(i)}(u). \end{aligned}$$

On the other hand, if $r \in Q_0$ (in which case all successor configurations $r'(j')$ must satisfy $r' \in C$) or $r \in Q_1$ and π selects transition that stays in C , then we have

$$\begin{aligned} \mathbb{E}^{\pi}(\hat{m}^{(i+1)} \mid \text{Run}(u)) &= \mathbb{E}^{\pi}(C^{(i+1)} + \bar{z}_{S^{(i+1)}} - (i+1) \cdot \bar{x}_C \mid \text{Run}(u)) \\ &= C^{(i)}(u) + \mathbb{E}^{\pi}(C^{(i+1)} - C^{(i)} - \bar{x}_C + \bar{z}_{S^{(i+1)}} \mid \text{Run}(u)) - i \cdot \bar{x}_C \\ &= C^{(i)}(u) - \bar{x}_C + \underbrace{\sum_{(r,k,r') \in \delta} p_{r'(j')} \cdot (k + \bar{z}_{r'})}_{\geq \bar{z}_r \text{ since } (\bar{x}_C, (\bar{z}_q)_{q \in C}) \text{ is a solution of } \mathcal{L}} - i \cdot \bar{x}_C \\ &\geq C^{(i)}(u) + \bar{z}_{S^{(i)}(u)} - i \cdot \bar{x}_C = \hat{m}^{(i)}(u). \end{aligned}$$

Thus, $\{\hat{m}^{(i)}\}_{i \geq 0}$ is indeed a submartingale. It is easy to see that $\{\hat{m}^{(i)}\}_{i \geq 0}$ has bounded differences.

Clearly, the membership of every run ω in $\{T_{\rightarrow} \leq n\}$ depends only on finite prefix of ω of length n , and thus T_{\rightarrow} is a stopping time relative to the filtration $\{\mathcal{F}_i\}_{i \geq 0}$. Also,

for every run ω we have $T_{\rightarrow}(\omega) \leq T(\omega)$ and since we assume that $\mathbb{E}^\pi q(i) < \infty$, we must also have $\mathbb{E}^\pi T_{\rightarrow} < \infty$. Thus the Optional stopping theorem applies and we have $\mathbb{E}^\pi \hat{m}^{(0)} \leq \mathbb{E}^\pi \hat{m}^{(T_{\rightarrow})}$. But $\hat{m}^{(0)} = l + \bar{z}_q$ and $\hat{m}^{(T_{\rightarrow})} \leq \mathbb{E}^\pi C^{(T_{\rightarrow})} + \max_{r \in C} \bar{z}_r + 1 + |\bar{x}_C| \cdot \mathbb{E}^\pi T_{\rightarrow}$. This gives us $\mathbb{E}^\pi T_{\rightarrow} \geq (l + \bar{z}_q - \max_{r \in C} \bar{z}_r - 1 - \mathbb{E}^\pi C^{(T_{\rightarrow})}) / |\bar{x}_C| \geq (l - V_C - 1 - \mathbb{E}^\pi C^{(T_{\rightarrow})}) / |\bar{x}_C|$. \square

In the following we say that q is a MEC state of \mathcal{A} if it lies in some MEC of \mathcal{A} . Otherwise we say that q is a non-MEC state.

We call state q' a transient successor of state q if both q and q' are non-MEC states and q' is reachable from q along a path that doesn't visit any MEC. We denote $n_{\mathcal{A}}$ the maximal number of transient successors of any state in \mathcal{A} .

Lemma 21. *Let \mathcal{A} be arbitrary OC-MDP and let q be arbitrary state of \mathcal{A} that is not contained in any MEC. Then under arbitrary strategy π the probability that, when starting in q , we will reach some MEC of \mathcal{A} in at most $n_{\mathcal{A}}$ steps, is at least $p_{\min}^{n_{\mathcal{A}}}$.*

Proof. We inductively define sets $H_0, H_1, \dots \subseteq 2^{[Q]}$. We set $H_0 = \{q\}$. Then, we construct H_i from H_{i-1} by initially setting $H_i = \emptyset$ and then performing the following operation for every set $R \in H_{i-1}$: We find a state $q_R \in R$ such that q_R is not contained in any MEC of \mathcal{A} and $\{s \mid q_R \rightsquigarrow s\} \cap R = \emptyset$. If there is no such state in R , then we add R to H_i . Otherwise:

- If q_R is a stochastic state, then we set $R' = R \cup \{s \mid q_R \rightsquigarrow s\}$ and add R' to H_i .
- If q_R is a non-deterministic state, then we denote $\{s \mid q_R \rightsquigarrow s\} = \{s_1, \dots, s_n\}$. After this, we create n new sets R_1, \dots, R_n , where $R_i = R \cup \{s_i\}$. Finally, we add sets R_1, \dots, R_n to H_i .

For every i and every $R \in H_i$ all the non-MEC states in R are transient successors of q . Thus, $H_{n_{\mathcal{A}}} = H_{n_{\mathcal{A}}+1}$. We claim that every set $R \in H_{n_{\mathcal{A}}}$ must contain at least one MEC-state of \mathcal{A} . Assume, for the sake of contradiction, that there is some $R \in H_{n_{\mathcal{A}}}$ containing only non-MEC-states. Then R satisfies the following: for every state $q \in R$, if q is non-deterministic then there is at least one state $s \in R$ such that $q \rightsquigarrow s$; otherwise, if q is stochastic, then $\{s \mid q \rightsquigarrow s\} \subseteq R$. This also means, that restriction of \mathcal{A} to set R , i.e. the tuple $\mathcal{A}_R = (R, (R \cap Q_0, R \cap Q_1), \delta \cap (R \times \{+1, 0, -1\} \times R), \{P_q\}_{q \in R \cap Q_0})$, is again a OC-MDP. As every OC-MDP, the \mathcal{A}_R also contains at least one MEC E , which must be contained in some MEC of \mathcal{A} . This contradicts the assumption that R contains only non-MEC states.

Now let π be arbitrary strategy and $i \geq 0$. Denote $R_i(\pi)$ the set of states that are, when starting in q , reached under π in at most i steps. From the construction of H_i it follows by straightforward induction, that there is some set $R \in H_i$ such that $R \subseteq R_i(\pi)$. In particular, there is some set $R \in H_{n_{\mathcal{A}}}$ such that $R \subseteq R_{n_{\mathcal{A}}}(\pi)$. Since R must contain at least one MEC-state of \mathcal{A} , there is some history u of length at most $n_{\mathcal{A}}$ such that u reaches a MEC state and $\mathbb{P}_q^\pi(\text{Run}(u)) > 0$. Then clearly $\mathbb{P}_q^\pi(\text{Run}(u)) \geq p_{\min}^{n_{\mathcal{A}}}$ and this proves the lemma. \square

Corollary 22. *Let q be an arbitrary state of \mathcal{A} . Denote T_M the random variable on runs starting in q that returns the first point in time, when some MEC of \mathcal{A} is reached. Then for arbitrary strategy π and every $k \geq 1$ we have $\mathbb{P}_q^\pi(T_M \geq k) \leq 4d^k$, where $d = \exp(-p_{\min}^{n_{\mathcal{A}}}/n_{\mathcal{A}})$.*

Proof. If $p_{\min} = 1$ then $\mathbb{P}_q^\pi(T_M > n_{\mathcal{A}}) = 0$ and thus the Lemma trivially holds. Otherwise we have $p_{\min} \leq 1/2$. From the previous lemma we immediately see that $\mathbb{P}_q^\pi(T_M \geq k) \leq (1 - p_{\min}^{n_{\mathcal{A}}})^{\lfloor \frac{k-1}{n_{\mathcal{A}}} \rfloor}$. We can now compute

$$\mathbb{P}_q^\pi(T_M \geq k) \leq (1 - p_{\min}^{n_{\mathcal{A}}})^{\lfloor \frac{k-1}{n_{\mathcal{A}}} \rfloor} \leq (1 - p_{\min}^{n_{\mathcal{A}}})^{\frac{k}{n_{\mathcal{A}}} - 2} = \frac{(1 - p_{\min}^{n_{\mathcal{A}}})^{\frac{k}{n_{\mathcal{A}}}}}{(1 - p_{\min}^{n_{\mathcal{A}}})^2} \leq 4(1 - p_{\min}^{n_{\mathcal{A}}})^{\frac{k}{n_{\mathcal{A}}}} \leq 4d^k.$$

□

Let r and r' be two states of \mathcal{A} that lie in the same MEC C . Then clearly $t_r = t_{r'}$. We will denote t_C the common value t_r of all r in C .

We now prove Proposition 19 for MEC-acyclic OC-MDPs. We say that a OC-MDP \mathcal{A} is MEC-acyclic if there is no cycle in \mathcal{A} containing states from two different MECs. Equivalently, one can say that \mathcal{A} is MEC-acyclic if no run in \mathcal{A} returns to some MEC once it leaves this MEC. The *height* of a state q in MEC-acyclic OC-MDP \mathcal{A} , which we denote $\text{height}(q)$, is the maximal number of MECs visited by any path starting in q . The height of a given MEC C is the common height of all its states.

For any OC-MDP \mathcal{A} we denote $\|\mathcal{A}_{\max}\| = \max\{\|\mathcal{A}_C\| \mid C \in \text{MEC}(\mathcal{A})\}$.

Lemma 23. *Let \mathcal{A} be a MEC-acyclic OC-MDP. Then there is a number $K = \exp(\|\mathcal{A}_{\max}\|^{O(1)}) \cdot O(n_{\mathcal{A}}/p_{\min}^{n_{\mathcal{A}}})$ such that the following holds for every memoryless deterministic strategy π and every initial configuration $q(i)$:*

$$\mathbb{E}^\pi T \geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W. \quad (12)$$

Moreover, K is computable in time polynomial in $\|\mathcal{A}_{\max}\| \cdot \log(p_{\min}) \cdot n_{\mathcal{A}}$ by algorithm that takes as an input number $n_{\mathcal{A}}$ and set of strongly connected OC-MDPs $\{\mathcal{A}_C \mid C \in \text{MEC}(\mathcal{A})\}$.

Proof. Recall that we denote $d = \exp(-p_{\min}^{n_{\mathcal{A}}}/n_{\mathcal{A}})$ and set

$$K = \max \left\{ \frac{4}{(1-d)^2 \cdot |\bar{x}_0|}, \frac{1 + \max_{C \in \text{MEC}(\mathcal{A})} V_C}{|\bar{x}_0|} \right\}.$$

The asymptotic upper bound on K is easy to check, since numbers \bar{x}_0 and V_C for $C \in \text{MEC}(\mathcal{A})$ are computed by solving linear program \mathcal{L} for MECs of \mathcal{A} ; also recall that $1/(1-d)^2 \leq 4n_{\mathcal{A}}^2/p_{\min}^{2n_{\mathcal{A}}}$ by standard calculus computation. This also shows that K can be computed in time polynomial in $\|\mathcal{A}_{\max}\| \cdot \log(p_{\min}) \cdot n_{\mathcal{A}}$ if we know numbers $n_{\mathcal{A}}$ and p_{\min} and OC-MDPs \mathcal{A}_C for every MEC C of \mathcal{A} .

Note that in every OC-MDP we have $\mathbb{E}^\pi W < \infty$ under any strategy π (by Lemma 18). Therefore, both inequalities trivially hold if $\mathbb{E}^\pi q(i) = \infty$. From now on we will assume that $\mathbb{E}^\pi q(i) < \infty$. In particular, we assume that under π the configuration with zero counter is reached almost surely from $q(i)$. We proceed by induction on $\text{height}(q)$. For every height we will prove the inequality separately for q being a non-MEC state and MEC-state, respectively.

To start the induction, suppose that q lies in MEC C of height 1. But then there are no transitions leaving C . In particular, we have $\mathbb{E}^\pi W = 1$. From part (B1) of Proposition 5 and from $K \geq \frac{V_C}{|\bar{x}_0|}$ we have

$$\mathbb{E}^\pi q(i) \geq \frac{i - V_C}{|\bar{x}_C|} \geq \frac{i}{|t_q|} - K.$$

The second equality holds because for state q that lies in MEC C with no outgoing transitions we have $t_q = \bar{x}_C$.

Suppose now that q is a non-MEC-state of height h and that (12) holds for all MEC-states of height at most h .

Denote F_C the event that the first MEC encountered on a run is C . Note that all MECs with $\mathbb{P}_{q(i)}^\pi(F_C) > 0$ have height at most h . Denote D the union of all MECs C with $\mathbb{P}_{q(i)}^\pi(F_C) > 0$. Similarly to previous proofs we can write $\mathbb{E}^\pi q(i) = E^\pi(T_1 + T_2)$ where T_1 returns the first point in time when the run hits either D or a configuration with a zero counter and T_2 returns time to hit a configuration with a zero counter after hitting D (or 0, if the run terminates before hitting D or never hits D at all). Since both these random variables are non-negative, it suffices to prove the required bound (12) for $\mathbb{E}^\pi T_2$.

As in previous proofs, we use the notation D_m (for $m > 0$) for the set of all runs that do not terminate before reaching D and at the same time they reach D with counter value m . (Also recall that we denote D_0 set of runs that terminate before or in the exact moment of reaching D .) Moreover we denote D_m^C the event $F_C \cap D_m$. Finally, we denote

$$B(l, j, C) := \frac{j}{|t_C|} - K \cdot (\mathbb{E}^\pi(W \mid D_l^C) - 1).$$

$$\text{Clearly } \sum_{C \in \text{MEC}(\mathcal{A}), l \geq 0} \mathbb{P}_{q(i)}^\pi(D_l^C) = \sum_{C \in \text{MEC}(\mathcal{A})} \mathbb{P}_{q(i)}^\pi(F_C) = 1.$$

We have

$$\mathbb{E}^\pi T_2 = \sum_{C \in \text{MEC}(\mathcal{A})} \mathbb{P}_{q(i)}^\pi(F_C) \cdot \mathbb{E}^\pi(T_2 \mid F_C). \quad (13)$$

We can write

$$\mathbb{P}_{q(i)}^\pi(F_C) \cdot \mathbb{E}^\pi(T_2 \mid F_C) = \sum_{l=0}^{\infty} \mathbb{E}^\pi(T_2 \mid D_l^C) \cdot \mathbb{P}_{q(i)}^\pi(D_l^C). \quad (14)$$

By induction hypothesis we have for every $l \geq 0$

$$\mathbb{E}^\pi(T_2 \mid D_l^C) \geq \frac{l}{|t_C|} - K \cdot (\mathbb{E}^\pi(W \mid D_l^C) - 1) = B(l, l, C). \quad (15)$$

Especially for every $l \geq i$ we have

$$\mathbb{E}^\pi(T_2 \mid D_l^C) \geq \frac{i}{|t_C|} - K \cdot (\mathbb{E}^\pi(W \mid D_l^C) - 1) = B(l, i, C). \quad (16)$$

Further, if we denote $g_l = i - l$ then for $l < i$ we can write

$$\mathbb{E}^\pi(T_2 \mid D_l^C) \geq \frac{(i - g_l)}{|t_C|} - K \cdot (\mathbb{E}^\pi(W \mid D_l^C) - 1) = B(l, i, C) - \frac{g_l}{|t_C|}. \quad (17)$$

We can now plug (16) and (17) into (14) and compute

$$\begin{aligned}
\mathbb{E}^\pi T_2 &\geq \sum_{C \in \text{MEC}(\mathcal{A})} \left(\sum_{l=i}^{\infty} (B(l, i, C) \cdot \mathbb{P}_{q(i)}^\pi(D_l^C)) + \sum_{l=0}^{i-1} \left(\left(B(l, i, C) - \frac{g_l}{|t_C|} \right) \cdot \mathbb{P}_{q(i)}^\pi(D_l^C) \right) \right) \\
&= \sum_{C \in \text{MEC}(\mathcal{A})} \left(\sum_{l=0}^{\infty} (B(l, i, C) \cdot \mathbb{P}_{q(i)}^\pi(D_l^C)) - \sum_{l=0}^{i-1} \frac{g_l}{|t_C|} \cdot \mathbb{P}_{q(i)}^\pi(D_l^C) \right) \\
&= i \cdot \underbrace{\sum_{C \in \text{MEC}(\mathcal{A})} \left(\frac{\mathbb{P}_{q(i)}^\pi(F_C)}{|t_C|} \right)}_{\geq \frac{1}{|t_q|}} - K \cdot \sum_{\substack{C \in \text{MEC}(\mathcal{A}), \\ l \geq 0}} (\mathbb{E}^\pi(W \mid D_l^C) - 1) \cdot \mathbb{P}_{q(i)}^\pi(D_l^C) \\
&\quad - \sum_{C \in \text{MEC}(\mathcal{A})} \sum_{l=0}^{i-1} \left(\frac{g_l}{|t_C|} \cdot \mathbb{P}_{q(i)}^\pi(D_l^C) \right) \\
&\geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W + K - \sum_{C \in \text{MEC}(\mathcal{A})} \sum_{l=0}^{i-1} \left(\frac{g_l}{|\bar{x}_0|} \cdot \mathbb{P}_{q(i)}^\pi(D_l^C) \right) \\
&= \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W + K - \frac{\sum_{l=0}^{i-1} (g_l \cdot \mathbb{P}_{q(i)}^\pi(D_l))}{|\bar{x}_0|}. \tag{18}
\end{aligned}$$

From Corollary 22 we have

$$\frac{\sum_{l=0}^{i-1} (g_l \cdot \mathbb{P}_{q(i)}^\pi(D_l))}{|\bar{x}_0|} \leq \frac{4 \cdot \sum_{l=0}^{i-1} g_l d^{g_l}}{|\bar{x}_0|} = \frac{4 \cdot \sum_{g_l=1}^i g_l d^{g_l}}{|\bar{x}_0|} \leq \frac{4}{(1-d)^2} \leq K, \tag{19}$$

since no run in D_l , for $l < i$, can hit D in less than g_l steps.

This gives us $K - \sum_{l=0}^{i-1} \frac{g_l}{|\bar{x}_0|} \cdot \mathbb{P}_{q(i)}^\pi(D_l) \geq 0$ and together with (18) we have

$$\mathbb{E}^\pi T_2 \geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W,$$

which proves that (12) holds for q .

Suppose now that q lies in MEC C of height h and that (12) holds for all states of height $h - 1$. The inequality (12) especially holds for all states $q' \in Q_{fin} \setminus C$ such that there is a transition from p to q' for some $p \in C$. We will call every such state q' a *C-gate* and denote $G(C)$ the set of all C -gates. From the definition of t_q it follows that $\frac{1}{|t_q|} \leq \frac{1}{|\bar{x}_C|}$ and $\frac{1}{|t_q|} \leq \frac{1}{|t_{q'}|}$ for any C -gate q' .

We can again express T as a sum of T_1 and T_2 , where T_1 returns the first point in time when the run visits configuration $r(l)$ with either $r \notin C$ or $l = 0$, and T_2 returns time to visit a configuration with a zero counter after leaving C (or 0, if the run terminates before leaving C or never leaves C – formally we again have $T_2(\omega) = -T_1(\omega) + T(\omega)$ if $T_1(\omega) < \infty$ and $T_2(\omega) = 0$ otherwise). From Lemma 20 we have

$$\mathbb{E}^\pi(T_1) \geq \frac{i - V_C - 1 - \mathbb{E}^\pi C^{(T_1)}}{|\bar{x}_C|} \geq \frac{i - \mathbb{E}^\pi C^{(T_1)}}{|t_q|} - \frac{V_C + 1}{|\bar{x}_0|}. \tag{20}$$

Now consider T_2 . For state q' not contained in C we denote $F_l^{q'}$ the set of all runs ω that visit configuration $q'(l)$ when they leave C for the first time, i.e. $\omega \in F_l^{q'}$ iff $S^{(T_1)}(\omega) = q'$ and $C^{(T_1)}(\omega) = l$. Note that for every q' such that $\mathbb{P}_{q(i)}^\pi(F_l^{q'}) > 0$ we must have $q' \in G(C)$. If we denote $F_l = \bigcup_{q' \in G(C)} F_l^{q'}$, then it is easy to see that $\mathbb{E}^\pi C^{(T_1)} = \sum_{l \in \mathbb{N}} l \cdot \mathbb{P}_{q(i)}^\pi(F_l)$. Finally, denote lv_C the event that the run leaves C at least once (i.e. $\omega \in lv_C$ iff $\omega \in D_l^{q'}$ for some l and q'). We have

$$\begin{aligned}
\mathbb{E}^\pi T_2 &= \sum_{\substack{q' \in G(C), \\ l \geq 0}} \mathbb{E}^\pi(T_2 \mid F_l^{q'}) \cdot \mathbb{P}_{q(i)}^\pi(F_l^{q'}) \\
&\geq \sum_{\substack{q' \in G(C), \\ l \geq 0}} \left(\left(\frac{l}{|t_{q'}|} - K \cdot (\mathbb{E}^\pi(W \mid F_l^{q'}) - 1) \right) \cdot \mathbb{P}_{q(i)}^\pi(F_l^{q'}) \right) \\
&\geq \sum_{\substack{q' \in G(C), \\ l \geq 0}} \left(\frac{l}{|t_q|} - K \cdot (\mathbb{E}^\pi(W \mid F_l^{q'}) - 1) \right) \cdot \mathbb{P}_{q(i)}^\pi(F_l^{q'}) \\
&= \frac{\sum_{l \geq 0} (l \cdot \mathbb{P}_{q(i)}^\pi(F_l))}{|t_q|} - K \cdot (\mathbb{E}^\pi(W \mid lv_C) \cdot \mathbb{P}_{q(i)}^\pi(lv_C) - \mathbb{P}_{q(i)}^\pi(lv_C)) \\
&= \frac{\mathbb{E}^\pi C^{(T_1)}}{|t_q|} - K \cdot (\mathbb{E}^\pi(W \mid lv_C) \cdot \mathbb{P}_{q(i)}^\pi(lv_C) - \mathbb{P}_{q(i)}^\pi(lv_C)), \tag{21}
\end{aligned}$$

where the inequality on the second line follows from induction hypothesis.

Denote $\overline{lv_C}$ the complement of lv_C . We trivially have $\mathbb{E}^\pi(W \mid \overline{lv_C}) \geq 1$. Putting (20) and (21) together we obtain

$$\begin{aligned}
\mathbb{E}^\pi q(i) &\geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi(W \mid lv_C) \cdot \mathbb{P}_{q(i)}^\pi(lv_C) + K \cdot \mathbb{P}_{q(i)}^\pi(lv_C) - \underbrace{\frac{V_C + 1}{|\bar{x}_0|}}_{\leq K} \\
&\geq \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi(W \mid lv_C) \cdot \mathbb{P}_{q(i)}^\pi(lv_C) - K \cdot 1 \cdot (1 - \mathbb{P}_{q(i)}^\pi(lv_C)) \\
&\geq \frac{i}{|t_q|} - K \cdot (\mathbb{E}^\pi(W \mid lv_C) \cdot \mathbb{P}_{q(i)}^\pi(lv_C) + \mathbb{E}^\pi(W \mid \overline{lv_C}) \cdot \mathbb{P}_{q(i)}^\pi(\overline{lv_C})) \\
&= \frac{i}{|t_q|} - K \cdot \mathbb{E}^\pi W.
\end{aligned}$$

Thus, (12) indeed holds for q . \square

We will now finish the proof of Proposition 19 for arbitrary OC-MDP with $Q = Q_{fin}$.

To achieve this, for arbitrary OC-MDP \mathcal{A} and any natural number k we define a new MEC-acyclic OC-MDP $\mathcal{A}(k)$ of height $k + 1$; we will augment states of \mathcal{A} with additional information, that will allow us to remember number of visits of MECs. Once we know that we have left a MEC for the k -th time, we allow to switch to a new state with a counter-decreasing self-loop. To be more specific, call the transition $q \rightsquigarrow q'$ a *crossing*,

if there exists a MEC C such that $q \in C$, $q' \notin C$. Then for $\mathcal{A} = (Q, (Q_0, Q_1), \delta, P)$ we set $\mathcal{A}(k) = (Q^k, (Q_0^k, Q_1^k), \delta^k, P^k)$, where $Q^k = \{(q, l) \mid q \in Q, 1 \leq l \leq k\} \cup \{\perp\}$, and

$$\begin{aligned} \delta^k = & \{((q, l), i, (q', l)) \mid (q, i, q') \in \delta \text{ and } (q, i, q') \text{ is not a crossing}\} \\ & \cup \{((q, l), i, (q', l-1)) \mid l > 1, (q, i, q') \in \delta \text{ is a crossing}\} \\ & \cup \{((q, 1), i, \perp) \mid \exists q' \text{ such that } (q, i, q') \in \delta \text{ is a crossing}\} \\ & \cup \{(\perp, -1, \perp)\}. \end{aligned}$$

Partition of states (Q_0^k, Q_1^k) and probability distribution P^k is derived from \mathcal{A} in obvious way, we just specifically put $\perp \in Q_0^k$.

Slightly abusing the notation we denote t_{q^k} the minimal trend achievable from state (q, k) in $\mathcal{A}(k)$.

For every deterministic strategy π in \mathcal{A} there is naturally corresponding deterministic strategy $\pi(k)$ in $\mathcal{A}(k)$, formally defined as follows: for any history $\bar{H} = (q_0, l_0)(j_0) \dots (q_m, l_m)(j_m)$ in $\mathcal{A}(k)$ we denote $q'(j')$ configuration of \mathcal{A} such that $\pi(q_0(j_0) \dots q_m(j_m))$ selects transition leading to configuration $q'(j')$; then we define $(\pi(k))(H)$ to select transition leading to configuration c of $\mathcal{A}(k)$ such that

$$c = \begin{cases} \perp(j') & \text{if } l_m = 1 \text{ and } q_m \rightsquigarrow q' \text{ is a crossing,} \\ (q', l_m - 1)(j') & \text{if } l_m > 1 \text{ and } q_m \rightsquigarrow q' \text{ is a crossing,} \\ (q', l_m)(j') & \text{otherwise.} \end{cases}$$

To differentiate between computations in \mathcal{A} and $\mathcal{A}(k)$, we again slightly abuse notation and denote $\mathbb{P}^{\pi(k)}$ and $\mathbb{E}^{\pi(k)}$ the probability and expected value, respectively, computed in $\mathcal{A}(k)$ under strategy $\pi(k)$. Note that if π is memoryless deterministic, then $\pi(k)$ is also memoryless deterministic.

It is clear that for any strategy π in \mathcal{A} and any $k \geq 1$ we have $\mathbb{E}^\pi q(i) \geq \mathbb{E}^{\pi(k)}(q, k)(i)$. We can thus use the Lemma 23 to show that for any memoryless deterministic strategy π and any $k \geq 1$ we have

$$\mathbb{E}^\pi q(i) \geq \frac{i}{|t_{q^k}|} - K \cdot \mathbb{E}_{(q,k)(i)}^{\pi(k)} W, \quad (22)$$

for a suitable number K . Note that for every k the MECs of $\mathcal{A}(k)$ are exactly copies of MECs of \mathcal{A} (with the exception of MEC $\{\perp\}$). It is also easy to see that $n_{\mathcal{A}(k)} \leq |Q|$, for every k , and that p_{\min} is the same in \mathcal{A} and $\mathcal{A}(k)$ for every k . By Lemma 23 this means that $K \in \exp(\|\mathcal{A}\|^{O(1)})$ can be chosen the same for every k and that it can be computed by a polynomial-time algorithm that takes \mathcal{A} as its input. (This is important observation: we do not have to construct any MEC-acyclic OC-MDP in order to compute K .)

To finish the proof of Proposition 19 it suffices to show that

$$\lim_{k \rightarrow \infty} \left(\frac{i}{|t_{q^k}|} - K \cdot \mathbb{E}_{(q,k)(i)}^{\pi(k)} W \right) \geq \frac{i}{|t_q|} - K \cdot \mathbb{E}_{q(i)}^\pi W.$$

This is done in following two lemmas.

Lemma 24. *We have $\lim_{k \rightarrow \infty} \frac{1}{|t_{q^k}|} = \frac{1}{|t_q|}$.*

Proof. For any $k \geq 1$ we clearly have $|t_q|^{-1} \geq |t_{q^k}|^{-1}$, so it suffices to prove that $\lim_{k \rightarrow \infty} \frac{1}{|t_{q^k}|} \geq \frac{1}{|t_q|}$. Fix arbitrary $k \geq 1$.

Consider the “fast” counterless strategy ρ^k from Proposition 9, that realizes the minimal trend t_{q^k} in $\mathcal{A}(k)$. We define a new strategy ρ' in \mathcal{A} as follows: Initially, ρ' behaves exactly as ρ^k , simply omitting the information on current depth stored in states of $\mathcal{A}(k)$. When strategy ρ^k prescribes to switch to state \perp , the strategy ρ' starts to behave as the “fast” counterless strategy σ in \mathcal{A} from Proposition 9.

Denote $hit_k(\perp)$ the event that run in $\mathcal{A}(k)$ reaches state \perp . Simple computation, which uses the fact that, apart from $\{\perp\}$, all MECs of $\mathcal{A}(k)$ are copies of MECs of \mathcal{A} , reveals that

$$\underbrace{\sum_{C \in MEC(\mathcal{A})} \frac{\mathbb{P}_{q(i)}^{\rho'}(M_C)}{|\bar{x}_C|}}_{\geq \frac{1}{|t_q|}} - \underbrace{\sum_{C \in MEC(\mathcal{A}(k))} \frac{\mathbb{P}_{(q,k)(i)}^{\rho^k}(M_C)}{|\bar{x}_C|}}_{= \frac{1}{|t_{q^k}|}} \leq \mathbb{P}_{(q,k)(i)}^{\rho^k}(hit_k(\perp)) \cdot \left(\frac{1}{|\bar{x}_0|} - 1 \right).$$

From the construction of $\mathcal{A}(k)$ it easily follows that $\mathbb{P}_{(q,k)}^{\rho^k}(hit_k(\perp)) \leq \mathbb{P}_q^{\rho'}(W \geq k)$. By Lemma 18 we have that $\mathbb{P}_q^{\rho'}(W \geq k) \rightarrow 0$ as $k \rightarrow \infty$. This gives us

$$\frac{1}{|t_q|} - \lim_{k \rightarrow \infty} \frac{1}{|t_{q^k}|} \leq 0,$$

which proves the lemma. \square

Lemma 25. We have $\lim_{k \rightarrow \infty} \mathbb{E}_{(q,k)(i)}^{\pi(k)} W = \mathbb{E}_{q(i)}^{\pi} W$.

Proof. Fix arbitrary $k \geq 1$. We have $\mathbb{E}_{(q,k)(i)}^{\pi(k)} W = \sum_{l \geq 1} l \cdot \mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l)$ and $\mathbb{E}_{q(i)}^{\pi} W = \sum_{l \geq 1} l \cdot \mathbb{P}_{q(i)}^{\pi}(W = l)$. From the construction of $\mathcal{A}(k)$ it easily follows that for all $l \leq k$ we have $\mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l) = \mathbb{P}_{q(i)}^{\pi}(W = l)$ and thus

$$\begin{aligned} |\mathbb{E}_{q(i)}^{\pi} W - \mathbb{E}_{(q,k)(i)}^{\pi(k)} W| &\leq \sum_{l=k}^{\infty} l \cdot |\mathbb{P}_{q(i)}^{\pi}(W = l) - \mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l)| \\ &\leq \sum_{l=k}^{\infty} l \cdot \mathbb{P}_{q(i)}^{\pi}(W = l) + \sum_{l=k}^{\infty} l \cdot \mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l). \end{aligned}$$

From Lemma 18 we have that $\mathbb{P}_{q(i)}^{\pi}(W = l) \leq b \cdot c^l$ for suitable numbers b and $0 < c < 1$. Moreover, $\mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l) = 0$ for all $l \geq 2 \cdot (k + 1)$. Also, since $\mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l) = \mathbb{P}_{q(i)}^{\pi}(W = l)$ for $l \leq k$, we have $\sum_{l=k}^{\infty} \mathbb{P}_{(q,k)(i)}^{\pi(k)}(W = l) = \sum_{l=k}^{\infty} \mathbb{P}_{q(i)}^{\pi}(W = l) \leq b \cdot \sum_{l=k}^{\infty} c^l$. Thus we can write $|\mathbb{E}_{q(i)}^{\pi} W - \mathbb{E}_{(q,k)(i)}^{\pi(k)} W| \leq b \cdot \sum_{l=k}^{\infty} c^l + 2 \cdot (k + 1) \cdot b \cdot \sum_{l=k}^{\infty} c^l \leq 3 \cdot (k + 1) \cdot b \cdot \sum_{l=k}^{\infty} c^l$. From standard results on power series we know that

$$\lim_{k \rightarrow \infty} (k + 1) \cdot \sum_{l=k}^{\infty} c^l \leq \lim_{k \rightarrow \infty} \sum_{l=k}^{\infty} (l + 1) \cdot c^l = 0$$

and thus also $\lim_{k \rightarrow \infty} |\mathbb{E}_{q(i)}^{\pi} W - \mathbb{E}_{(q,k)(i)}^{\pi(k)} W| = 0$. This proves the lemma. \square

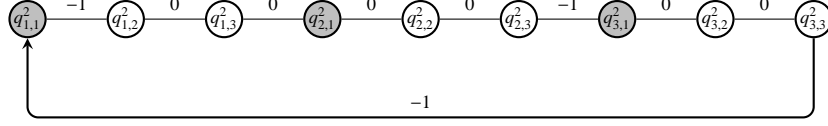


Fig. 2. The gadget for x_2 when $n = 2$. Shadow states are the entry points.

A.4 Proofs of Section 4

Lemma 26. *Given a propositional formula φ in CNF, one can compute a OC-MDP \mathcal{A} , a configuration $p(K)$ of \mathcal{A} , and a number N in time polynomial in $\|\varphi\|$ such that*

- $N \leq |Q| \cdot K$, where Q is the set of control states of \mathcal{A} ;
- if φ is satisfiable, then $\text{Val}(p(K)) = N - 1$;
- if φ is not satisfiable, then $\text{Val}(p(K)) = N$.

Proof. Let $\varphi \equiv C_1 \wedge \dots \wedge C_n$ where C_1, \dots, C_n are clauses over propositional variables x_1, \dots, x_m . We may safely assume that $n \geq 5$. Let π_1, \dots, π_m be the first m prime numbers. For every x_i , where $1 \leq i \leq m$, we construct the gadget shown in Fig. 2. That is, we fix $\pi_i \cdot (n + 1)$ fresh stochastic control states $q_{j,\ell}^i$, where $1 \leq j \leq \pi_i$ and $1 \leq \ell \leq n + 1$, and connect them by transitions in the following way:

- $q_{1,1}^i \xrightarrow{-1} q_{1,2}^i$, $q_{1,\ell}^i \xrightarrow{0} q_{1,\ell+1}^i$ for all $2 \leq \ell \leq n$, $q_{1,n+1}^i \xrightarrow{0} q_{2,1}^i$;
- for all $2 \leq j \leq \pi_i$ we include the following transitions:
 - $q_{j,\ell}^i \xrightarrow{0} q_{j,\ell+1}^i$ for all $1 \leq \ell \leq n$,
 - $q_{j,n+1}^i \xrightarrow{-1} q_{j',1}^i$, where j' is either $j + 1$ or 1 depending on whether $j < \pi_i$ or not, respectively.

Since each $q_{j,\ell}^i$ has exactly one successor, all of the above transitions have probability one. Also note that the total size of the constructed gadgets is polynomial in $\|\varphi\|$ because $\sum_{i=1}^m \pi_i$ is $O(m^2 \log m)$ (see, e.g., [2]).

The control states of the form $q_{j,1}^i$, where $1 \leq j \leq \pi_i$, are called the *entry points* for x_i . Note that in $q_{j,1}^i$, the counter is decremented in just one transition, while in the other entry points we need $n + 1$ transitions to decrement the counter.

An important technical observation about the entry points is the following: For every $k \geq 1$ and $1 \leq i \leq n$, there is exactly one *optimal* entry point $q_{j,1}^i$ such that $\text{Val}(q_{j,1}^i(k)) = k(n + 1) - n$, and for the other entry points $q_{j',1}^i$ we have that $\text{Val}(q_{j',1}^i(k)) = k(n + 1)$. To see this, consider the (unique) k' such that $1 \leq k' \leq \pi_i$ and $k = k' + c \cdot \pi_i$ for some $c \geq 0$. We put $j = 1$ if $k' = 1$, otherwise $j = \pi_i - k' + 2$. Now one can easily verify (with the help of Fig. 2) that $\text{Val}(q_{j,1}^i(k)) = k(n + 1) - n$, and $\text{Val}(q_{j',1}^i(k)) = k(n + 1)$ for the other entry points $q_{j',1}^i$.

Every $k \geq 1$ encodes a unique assignment $v_k : \{x_1, \dots, x_m\} \rightarrow \{\text{true}, \text{false}\}$ defined as follows: For every $1 \leq i \leq m$ we put $v_k(x_i) = \text{true}$ iff $q_{1,1}^i$ is the optimal entry point for k . Also observe that for every assignment $v : \{x_1, \dots, x_m\} \rightarrow \{\text{true}, \text{false}\}$ there is some $k \leq \prod_{i=1}^m \pi_i$ such that $v = v_k$.

We proceed by encoding the structure of C_1, \dots, C_n . For each clause $C_\ell \equiv y_{i_1} \vee \dots \vee y_{i_t}$, where every y_{i_h} is either x_{i_h} or $\neg x_{i_h}$, we fix a fresh non-deterministic control state c_ℓ and add the following transitions for every $1 \leq h \leq t$:

- if $y_{i_h} \equiv x_{i_h}$, then we add a transition $c_\ell \xrightarrow{0} q_{1,1}^{i_h}$;
- if $y_{i_h} \equiv \neg x_{i_h}$, then we add a transition $c_\ell \xrightarrow{0} q_{j,1}^{i_h}$ for every $2 \leq j \leq \pi_{i_h}$.

Using the definition of v_k and the above observation about the entry points, we immediately obtain that, for all $1 \leq \ell \leq n$ and $k > 1$,

- $v_k(C_\ell) = \text{true}$ iff $\text{Val}(c_\ell(k)) = k(n+1) - n + 1$;
- $v_k(C_\ell) = \text{false}$ iff $\text{Val}(c_\ell(k)) = k(n+1) + 1$.

Now, we add a fresh stochastic control state q_φ such that $q_\varphi \xrightarrow{0} c_\ell$ for every $1 \leq \ell \leq n$. The probability of each of these transitions is $1/n$. For every $k \geq 1$ we have that

- if $v_k(C_\ell) = \text{true}$, then $\text{Val}(q_\varphi(k)) = k(n+1) - n + 2$;
- if $v_k(C_\ell) = \text{false}$, then at least one clause is false, which implies

$$\text{Val}(q_\varphi(k)) \geq \frac{n-1}{n} \left(k(n+1) - n + 2 \right) + \frac{1}{n} \left(k(n+1) + 2 \right) = k(n+1) - n + 3.$$

The construction of \mathcal{A} is completed by adding a non-deterministic control state p and a family of stochastic control states d_1, \dots, d_n , where the transitions are defined as follows (here we need that $n \geq 5$):

- $p \xrightarrow{0} c_\varphi$, $p \xrightarrow{0} d_1$,
- $d_4 \xrightarrow{-1} d_5$, $d_n \xrightarrow{0} p$,
- $d_j \xrightarrow{0} d_{j+1}$ for all $1 \leq j < n$, $j \neq 4$.

Let σ be a pure memoryless strategy in $\mathcal{M}_{\mathcal{A}}^\infty$ such that

- in every configuration of the form $c_\ell(k)$, the strategy σ selects a transition to some optimal entry point for k . If all transitions lead to non-optimal entry points, any of them can be selected;
- in a configuration of the form $p(k)$, the strategy σ selects either the transition leading to $q_\varphi(k)$ or the transition leading to $d_1(k)$, depending on whether $v_k(\varphi) = \text{true}$ or not, respectively.

Obviously, σ is optimal in all configurations of the form $c_\ell(k)$, and hence it is also optimal in all configurations of the form $q_\varphi(k)$. By induction on k , we show that σ is optimal in $p(k)$, and $\text{Val}(p(k))$ equals either $k(n+1) - n + 3$ or $k(n+1) - n + 4$, depending on whether $v_{k'}(\varphi) = \text{true}$ for some $1 \leq k' \leq k$ or not, respectively.

- **k = 1.** If $v_1(\varphi) = \text{true}$, then $\mathbb{E}^\sigma p(1) = 4$. Further, it cannot be that $\mathbb{E}^{\sigma'} p(1) < 4$ for any pure strategy σ' , because
 - if σ' selects the transition from $p(1)$ to $d_1(1)$, then inevitably $\mathbb{E}^{\sigma'} p(1) = 5$;
 - if σ' selects the transition from $p(1)$ to $q_\varphi(1)$, then $\mathbb{E}^{\sigma'} p(1)$ cannot be less than 4 because σ plays optimally in $q_\varphi(1)$.

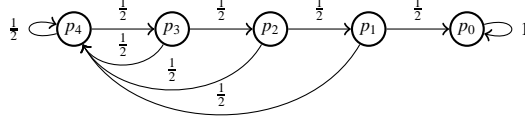


Fig. 3. The example gadget \mathcal{G}_4 .

If $v_1(\varphi) = \text{false}$, then $\mathbb{E}^\sigma p(1) = 5$, and this outcome cannot be improved by playing the transition from $p(1)$ to $q_\varphi(1)$ because σ is optimal in $q_\varphi(1)$ and $\mathbb{E}^\sigma q_\varphi(1) \geq 4$. Hence, σ is optimal in $p(1)$ and $\text{Val}(p(1))$ is either 4 or 5 depending whether $v_1(\varphi) = \text{true}$ or not, respectively.

- **Induction step.** Let us consider a configuration $p(k+1)$. If $v_{k+1}(\varphi) = \text{true}$, then $\mathbb{E}^\sigma p(k+1) = (k+1)(n+1) - n + 3$. Since σ plays optimally in $q_\varphi(k+1)$, this outcome cannot be improved by any pure strategy σ' which selects the transition from $p(k+1)$ to $q_\varphi(k+1)$. If σ' selects the transition from $p(k+1)$ to $d_1(k+1)$, then $p(k)$ is inevitably reached in exactly $n + 1$ transitions. By induction hypothesis, this leads to the outcome at least $(n+1) + k(n+1) - n + 3 = (k+1)(n+1) - n + 3$. Hence, σ is optimal and $\text{Val}(p(k+1)) = (k+1)(n+1) - n + 3$.

If $v_{k+1}(\varphi) = \text{false}$, then (by applying induction hypothesis) $\mathbb{E}^\sigma p(k+1)$ is equal either to $(n+1) + k(n+1) - n + 3$ or to $(n+1) + k(n+1) - n + 4$, depending on whether $v_{k'}(\varphi) = \text{true}$ for some $1 \leq k' \leq k$ or not, respectively. In both cases, this yields the desired outcome which cannot be improved by using the transition from $p(k+1)$ to $q_\varphi(k+1)$, because then the outcome is inevitably at least $(k+1)(n+1) - n + 4$.

Now, it suffices to put $K = \prod_{i=1}^m \pi_i$ and $N = K(n+1) - n + 4$. Since π_i is $O(i \log(i))$, the encoding size of \mathcal{A} is polynomial in $\|\varphi\|$, and the length of the binary encoding of K and N is also polynomial in $\|\varphi\|$. \square

By Lemma 26, the existence of an algorithm which computes $\text{Val}(p(k))$ up to an *absolute* error strictly less than $1/2$ in time $O(f)$ implies the existence of an algorithm for SAT and UNSAT whose time complexity is $O(f \circ p)$, where p is a polynomial. The same can be said about an algorithm which computes $\text{Val}(p(k))$ up to a *relative* error strictly less than $1/(2 \cdot |Q| \cdot k)$, where Q is the set of control states of \mathcal{A} . Also note that stochastic states in \mathcal{A} have outgoing edges whose probability is 1 or $1/n$, but it is trivial to modify the construction so that all of these probabilities are equal to $1/2$. So, Lemma 26 proves Theorem 10 for configurations of the form $q(i)$. Now we show that we can even take $i = 1$.

Let us consider the following OC-MDP \mathcal{G}_k : the set of control states is $\{p_0, \dots, p_k\}$, all of these states are stochastic, and there a transition from p_i to p_{i-1} and p_k for all $i \geq 1$. All transitions increment the counter by 1 and have probability $\frac{1}{2}$. The state p_0 is a dead-end with a self-loop. An example for $k = 4$ is given in Figure 3.

Lemma 27. *With probability higher than $\frac{1}{4}$, a run initiated in $p_k(1)$ visits a configuration $p_0(i)$ where $i \geq 2^k$.*

Proof. Notice that the probability of terminating in one step is less or equal to 2^{-k} , because in order to reach p_0 from p_k the process has to take a sequence of k transitions, as otherwise it restarts at p_k . Therefore, the probability that the process does not reach p_0 in i steps is greater or equal to $(1 - 2^{-k})^i$. For $i = 2^k$ we have that this value is $(1 - 2^{-k})^{2^k}$, but it is well-known that the sequence $(1 - \frac{1}{n})^n$ is increasing in n and converges to $\frac{1}{e}$. As for $n = 2$ this expression is equal $\frac{1}{4}$, for $k \geq 1$ we get that the probability of visiting p_0 with the counter value higher than 2^k is at least $\frac{1}{4}$. \square

We also need the following lemma:

Lemma 28. $\prod_{i=1}^m \pi_i \leq 2^{m^2}$, where π_m is the m -th smallest prime number.

Proof. Of course $\pi_1 = 2$. Bertrand's postulate states that for every $k > 1$ there is at least one prime number p such that $k < p < 2k$. From this we know that there is at least one prime in the following disjoint intervals $(2, 4)$, $(4, 8)$, $(8, 16)$, \dots which gives us an estimate on the $\pi_i \leq 2^i$. Therefore, $\prod_{i=1}^m \pi_i \leq \prod_{i=1}^m 2^i = 2^{m(m+1)/2} \leq 2^{m^2}$ for all $m \geq 1$. \square

With the help of Lemma 27 and Lemma 28, we can now prove the following:

Lemma 29. *Given a propositional formula φ in CNF, one can compute a OC-MDP \mathcal{B} that uses only probabilities $\frac{1}{2}$ on transitions such that being able to approximate $\text{Val}(q(1))$ up to the absolute error $\frac{1}{8}$ or the relative error $2^{-|Q|}$, where $|Q|$ is the number of control states of \mathcal{B} , suffices to establish whether φ is satisfiable or not.*

Proof. Let φ be an arbitrary CNF formula, we construct a polynomially sized OC-MDP \mathcal{B} with probabilities on transitions equal $\frac{1}{2}$, such that φ is not satisfiable iff the optimal termination time from one of the control states and counter value 1 is equal to $(n + 2)(2^{m^2+1} - 1) - 6$, where n and m are the number of clauses and variables in φ , respectively. We will build \mathcal{B} by combining the gadget \mathcal{G}_{m^2} (see Fig. 3), where m is the number of propositional variables in φ , with the OC-MDP \mathcal{A} that we obtain from Lemma 26 for φ . We let the initial state of \mathcal{B} be $p_{m^2}(1)$ and the initial control state p of \mathcal{A} replaces the control state p_0 in \mathcal{G}_{m^2} . Let x_k denote the probability that \mathcal{A} will be initiated at $p(k + 1)$ in \mathcal{B} , which is the same as saying that \mathcal{A} executes k transitions before reaching control state p . Of course $\sum_k x_k = 1$ and thanks to Lemma 27 we have $\sum_{k \geq 2^{m^2}} x_k > \frac{1}{4}$.

Assume that φ is not satisfiable. We know that the expected termination time from $p(k)$ in \mathcal{A} is equal to $k(n + 1) - n + 4$ for every k , where n is the number of clauses in φ . Therefore $\text{Val}(p_{m^2}(1)) = \sum_k x_k (k + k(n + 1) - n + 4)$. Let us consider a Markov chain M with positive rewards obtained from \mathcal{G}_{m^2} by ignoring the counter completely and assigning reward $n + 2$ to each transition. Notice that the expected total reward before M terminates is equal to $v := \sum_k x_k \cdot k(n + 2)$, so $\text{Val}(p_{m^2}(1)) - v = \sum_k x_k (n - 4) = n - 4$. It is quite straightforward to compute v to be $(n + 2)(2^{m^2+1} - 2)$, and so in the end get that $\text{Val}(p_{m^2}(1)) = (n + 2)(2^{m^2+1} - 1) - 6$.

Next, assume that φ is satisfiable. Let k' be the smallest number such that the assignment to the propositional variables corresponding to k' in the proof of Lemma 26 satisfies φ . We know that $k' \leq \prod_{i=1}^m \pi_i$ which is $\leq 2^{m^2}$ thanks to Lemma 28. We

also know that for all $k < k'$ we have $\text{Val}(p(k)) = k(n+1) - n + 4$ and for all $k \geq k'$ we have $\text{Val}(p(k)) = k(n+1) - n + 3$. Therefore in this case $\text{Val}(p_{m^2}(1)) = \sum_{k < k'} x_k (k(n+1) - n + 4) + \sum_{k \geq k'} x_k (k(n+1) - n + 3) = \sum_k x_k (k(n+1) - n + 4) - \sum_{k \geq k'} x_k \leq (n+2)(2^{m^2+1} - 1) - 6 - \sum_{k \geq 2^{m^2}} x_k \leq (n+2)(2^{m^2+1} - 1) - 6 - \frac{1}{4}$, where the last step follows from Lemma 27. Notice that the number of control states in \mathcal{B} is $|Q| \geq m^2 + \sum_m \pi_m(n+1)$, so $\frac{1}{8}((n+2)(2^{m^2+1} - 1) - 6) \leq 2^{-|Q|}$. \square